Estimating the Fractal Dimension and the Predictability of the Atmosphere

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ABSTRACT

The fractal dimension, Lyapunov-exponent spectrum, Kolmogorov entropy, and predictability are analyzed for chaotic attractors in the atmosphere by analyzing the time series of daily surface temperature and pressure over several regions of the United States and the North Atlantic Ocean with different climatic signal-to-noise ratios. Though the total number of data points (from about 13 800 to about 36 500) is larger than those used in previous studies, it is still too small to obtain a reliable estimate of the Grassberger–Procaccia correlation dimension because of the limitations discussed by Ruelle. However, it can be shown that this dimension is greater than 8. Also, it is pointed out that most, if not all, of the previous estimates of low fractal dimensions in the atmosphere are spurious. These results lead us to claim that there probably exist no low-dimensional strange attractors in the atmosphere. Because the fractal dimension has not yet been saturated, the Kolmogorov entropy and the error-doubling time obtained by the method of Grassberger and Procaccia are sensitive to the selection of the time delay and are thus unreliable. Geographic variability of the fractal dimension is suggested, but further verification is needed.

A practical and more reliable method for estimating the Kolmogorov entropy and error-doubling time involves the computation of the Lyapunov-exponent spectrum using the algorithm of Zeng et al. Using this method, it is found that the error-doubling time is about 2–3 days in Fort Collins, Colorado, about 4–5 days in Los Angeles, California, and about 5–8 days in the North Atlantic Ocean. The predictability time is longer over regions with a higher climatic signal-to-noise ratio (e.g., Los Angeles), and the predictability time of summer and/or winter data is longer than for the entire year. The difference between these estimates of error-doubling time and estimates based on general circulation models (GCMs) is discussed. It is also mentioned that the computation of the Lyapunov exponents is slightly sensitive to the selection of the time delay, possibly because the fractal dimension is very high in the atmosphere. Such sensitivity has not been mentioned in previous similar studies.

1. Introduction

Nonlinear phenomena occur in nature in many apparently different contexts, but they often display common features or can be understood using similar concepts. Deterministic chaos and fractal structure in dissipative dynamical systems (e.g., the atmosphere) are among the most important nonlinear paradigms. In the past 10 yr or so, practical methods have been developed to reconstruct the phase space from experimental time series (Takens 1981), to compute the fractal dimension of an attractor (Grassberger and Procaccia 1983a), which characterizes its metric structure (giving rise to static, time-independent invariants), and to evaluate the spectrum of Lyapunov exponents (Zeng et al. 1991, 1992) and the Kolmogorov entropy (Grassberger and Procaccia 1983b), which are dynamic invariants describing details of the temporal evolution of the system. The fractal dimension of the attractor measures to what extent the dynamics fills the embedding phase space and provides a lower bound for the number of independent variables (degrees of freedom) necessary to model the time evolution of the system. For simple systems, this dimensional characterization also provides an upper bound via the Whitney embedding theorem (e.g., see Takens 1981). Moreover, as recommended by the ECMWF Workshop (1988), the evaluation of the dimensionality of atmospheric attractors may have an impact on the number of elements needed in a Monte Carlo ensemble forecast of the extended range. Lyapunov exponents are the average exponential rates of divergence or convergence of nearby orbits in phase space. Any system containing at least one positive Lyapunov exponent is defined to be chaotic, with the magnitude of this exponent reflecting the time scale on which system dynamics become unpredictable.

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Following the new developments and concepts mentioned above, Nicolis and Nicolis (1984) analyzed the time series of the isotope record of deep-sea cores and obtained a low dimensionality (between three and four) for the climate system. Subsequently, Fraedrich (1986, 1987), Essex et al. (1987), and Keppenne and Nicolis (1989) analyzed daily average data over eastern North America and western Europe and have likewise concluded the existence of low-dimensional attractors. Also, theories of deterministic chaos and fractal structure have been applied to data in the atmospheric boundary layer (Tsonis and Elsner 1988), the pulse of storm rainfall (Sharifi et al. 1990), and some special atmospheric systems, such as the Southern Oscillation (Hense 1987) and cyclone tracks (Fraedrich and Leslie 1989; Fraedrich et al. 1990). Using entire global fields of data rather than single-point time series, Pierrehumbert (1990) discussed the dimension of global atmospheric variability. On the other hand, there are still some doubts among researchers concerning strange attractors in the atmosphere (Pool 1989). Ruelle (1990) discussed limitations to the Grassberger–Procaccia algorithm and found that many published estimates of fractal dimensions are spurious. Lorenz (1991) discussed another possible reason for apparently finding low-dimensional attractors in the atmosphere.

The purpose of this paper is to evaluate the chaotic properties of the atmosphere from the time series of daily average variables observed over different regions of the United States and the North Atlantic Ocean. These regions have different climatic signal-to-noise ratios. Our results will be compared with those mentioned above and with the results of traditional statistical methods (Madden 1976; Madden and Shea 1978). One difference between our work and previous work is that we relate the estimates of predictability time to the climatic signal-to-noise ratios. The data are discussed in section 2. The reconstruction technique of the phase space in which the dynamics can be evaluated is presented in section 3. The fractal dimensions of weather attractors are discussed in section 4. The spectrum of Lyapunov exponents and the predictability of the atmosphere are discussed in section 5. The conclusions from this study are given in section 6.

2. The data

The data utilized in this study include the daily surface temperature (ST) over a period of 100 years (1 January 1889–31 January 1989) and the daily surface pressure (SP) over a period of 90 years (1 June 1889–31 December 1979, except the periods 1 August 1940–31 October 1940 and 1 September 1961–31 December 1961) observed in Fort Collins, Colorado. These data were provided by T. McKee and J. Kleist of the Colorado Climate Center. Also used are the surface temperatures over a period of 39 years (1 January 1947–31 December 1985) observed in Los Angeles, California; these data were provided by the National Center for Atmospheric Research (NCAR). There are a limited number of missing data (less than 1%) in the above time series, and linear interpolations in time were used to fill in for these missing data. The two stations of Fort Collins and Los Angeles in the United States were selected in this study based on the climatic signal-to-noise ratio (SNR), which is lower in Fort Collins and higher in Los Angeles (Madden and Shea 1978).

The sea surface temperature (SST) and sea surface pressure (SSP) of two regions (BOX 244: 20°–30°N, 300°–310°W; BOX 139: 50°–60°N, 330°–340°W) over the North Atlantic Ocean are also used, with the SNR being relatively lower in BOX 139 and higher in BOX 244 (Madden 1976). These data were obtained from the Comprehensive Ocean–Atmosphere Data Set (COADS) (Woodruff et al. 1987). COADS is widely accepted as the best dataset over the global ocean, with state-of-the-art data quality control. The Compressed Marine Reports (CMR5) of COADS are used to obtain daily average data. Since the number of global marine reports were relatively small during World War II, only the data from 1 January 1950–31 December 1987 are used. In addition, our computations show that there is a considerable number of missing daily observations for 2° × 2°, 2° × 10°, or 5° × 5° areas within BOX 244 and BOX 139. Therefore, only the 10° × 10° box (i.e., BOX 244 and BOX 139) is used to compute time series of SST and SSP, with a small number of missing data for which linear interpolations were used to fill in the voids. Since the observational data at different stations over the whole of western Europe seem to be derived from a single deterministic dynamical system (Keppenne and Nicolis 1989), it is not unreasonable to expect that the averaging of observations within a 10° × 10° box over an ocean can only slightly affect the computations of chaotic properties from the time series.

In an effort to minimize the effects of seasonal variations of temperature, a mean temperature is computed for each day by averaging temperatures over the record for the same day every year. These daily means are then subtracted from each daily value. For brevity, the term temperature is used to refer to the temperature perturbation in subsequent sections. The above procedure does not apply to the observational values of the pressure. The total number of data points is given in Table 3.

In this paper, we analyze not only daily data described above, but also winter/summer data (daily data of winter/summer seasons that last 120 days commencing on 1 November and 1 May, respectively).

3. Phase-space reconstruction from experimental data

The attractor of a dynamical system can easily be obtained if the nonlinearly coupled differential equations for the relevant variables of the system are known. However, in a natural system, this full information is unknown or is lacking, so that the attractor of the system is not a priori accessible. In the present paper there is only a time series of temperature or pressure at given locations, and the attractor must be reconstructed in an artificial phase space: the space spanned by the full set of its relevant variables (Packard et al. 1980; Takens 1981). Let $x_i = x(i\Delta t)$ ($i = 1, 2, \cdots, N$) represent the time series of temperature or pressure, where N is the total number of observations, and $\Delta t = 1$ day is the time interval between measurements. A k-dimensional phase space is then constructed by forming the vectors

$$\mathbf{x}_i = (x_i, x_{i+m}, \cdots, x_{i+(k-1)m}), \qquad (1)$$

where $\tau = m\Delta t$ is the time delay, with the integer m chosen appropriately. Further discussion on this and other reconstruction methods are given in Zeng et al. (1992).

For an infinite amount of noise-free data, the time delay τ can, in principle, be chosen almost arbitrarily (Takens 1981). However, when the data are limited in number and are noisy, the quality of the analysis depends on the value chosen for τ. Different methods have been suggested for obtaining τ.

For a two-dimensional reconstruction ($k = 2$) in the presence of experimental noise, if τ is too small, x_i and x_{i+m} will be indistinguishable, and all trajectories will appear to lie on the diagonal. To avoid this, the chosen τ should make x_i and x_{i+m} independent; that is, the trajectory should fill the space. Figure 1 depicts the time dependence of the surface temperature for the Fort Collins station and gives a two-dimensional view of the trajectory with $\tau = 3$ days (or $m = 3$). It is seen that the trajectory fills the entire space, suggesting that the data are independent for $\tau = 3$ days (and that the system is in a phase space of greater than two dimensions).

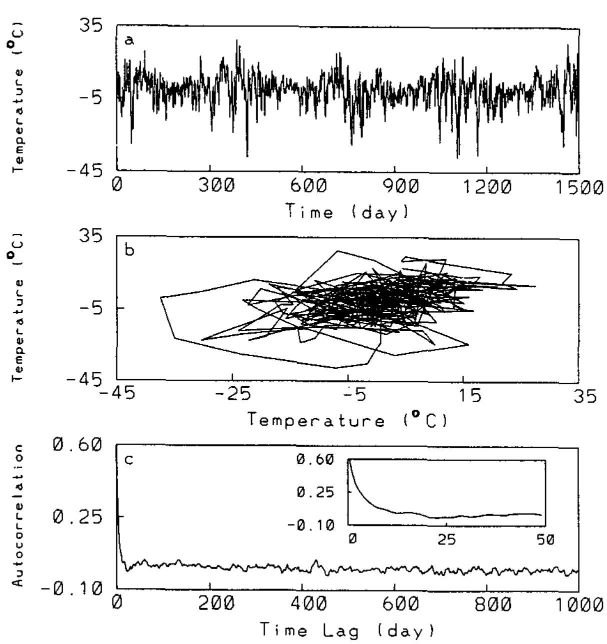

FIG. 1. (a) Time evolution of the surface temperature at Fort Collins, Colorado. (b) Time trajectory of the above time series evolving in a two-dimensional phase space of time-delay coordinates with $\tau = 3$ days. (c) Autocorrelation function of the above time series. The inset graph is a magnification of the region close to the origin in (c).

Another method to obtain τ, which guarantees linear independence, is to compute the autocorrelation function. Here τ may be defined as the lag time at which the autocorrelation function falls below a threshold value that is not uniquely defined. In general, this threshold value depends upon the problem and the assumptions about the dataset, but it is commonly taken as e^{-1} in meteorology, especially if the decay is nearly exponential (Zawadzki 1973). The autocorrelation function corresponding to the surface temperature at Fort Collins is given in Fig. 1c. It is seen that τ can be selected as 3 days. Using the above two methods, τ is selected in the range from 2 to 10 days for temperature or pressure data at the different locations, and these choices are shown in Table 1. These values also correspond with the characteristic times between independent estimates (Madden 1976; Madden and Shea 1978).

The value of τ can also be obtained by computing the mutual information (Fraser and Swinney 1986), which measures the general dependence, rather than linear dependence, of two variables. Although minimizing the mutual information was found to be a good criterion for the choice of time delay for a phase-portrait reconstruction from the time-series data of Fraser and Swinney (1986), this method has not yet been widely used, and it is not used in the paper.

4. Fractal dimensions of weather attractors

Dissipative dynamical systems (e.g., the atmosphere) that exhibit chaotic behavior often have an attractor in phase space that is strange. The existence of chaos and the properties of strange attractors can be verified and studied by examining the power spectra and computing fractal dimensions and Lyapunov exponents. Before the fractal dimensions are discussed, a brief discussion of power-spectrum analysis is presented.

Power-spectrum analysis is often used to qualitatively distinguish quasi-periodic or chaotic behavior from periodic structure and to identify different periods embedded in a chaotic signal (e.g., see Zeng et al. 1990). Chaos is characterized by a power spectrum of continuous appearance. Figure 2 shows such a power spectrum for the surface temperature at Fort Collins. There is no clear peak corresponding to the annual cycle, since this annual cycle has been removed before computation by subtracting the daily averages. However, such a peak does appear in the power spectrum of surface pressure data, which is not shown here. It is also seen from Fig. 2 and from the power spectra of all other analyzed data, that the spectra tend to resemble white noise beyond a cutoff frequency of about 0.4 cycles per day. This is consistent with the result of Keppenne and Nicolis (1989), which implies that a

TABLE 1. Dimensionality ν as a function of embedding dimension k for the analyzed data. All abbreviations are explained in section 2. No saturation values (ν_s) were obtained for some of the data, and this is indicated in the table by a horizontal line. The question mark after a value indicates it is probably spurious.

				k								
				5	7	9	11	13	15	17	19	
Location	Variable	τ (day)	Data					ν				ν_s
BOX 139	SST	5	daily	4.9	6.5	8.0	9.3	10.2	11.0	11.5	12.3	—
			winter	4.6	6.3	7.7	8.1	8.8	9.5	10.0	10.7	—
			summer	4.9	6.5	8.4	9.2	10.6	11.8	10.9	11.3	—
BOX 139	SSP	5	daily	4.9	6.7	8.0	9.1	9.4	9.5	9.6	9.7	9.6?
			winter	4.9	6.3	7.4	8.2	8.7	8.9	9.0	9.0	9.0?
			summer	4.9	6.4	7.7	8.6	8.9	8.9	8.8	9.0	8.9?
BOX 244	SST	5	daily	4.8	6.6	8.7	10.4	11.7	12.7	13.0	13.5	—
			winter	4.7	6.3	7.5	9.0	10.1	11.4	12.6	14.3	—
			summer	4.7	6.4	8.0	9.9	10.8	11.4	12.1	12.4	—
BOX 244	SSP	7	daily	4.8	6.5	8.3	9.2	9.5	10.2	10.2	10.0	10.1?
			winter	4.8	6.8	7.4	8.1	8.6	8.4	8.5	8.3	8.5?
			summer	4.8	6.5	7.7	8.8	9.5	10.0	10.0	9.9	10.0?
FCL	ST	3	daily	4.8	6.7	8.1	9.7	11.1	11.7	13.1	13.8	—
			winter	5.0	6.8	8.2	9.8	11.5	12.5	12.2	12.7	—
			summer	4.8	6.5	8.5	9.7	10.4	11.6	12.3	13.4	—
FCL	SP	2	daily	5.0	6.8	8.5	10.2	11.3	12.0	13.7	14.0	—
			winter	5.0	6.7	8.3	9.8	11.6	13.2	13.6	14.6	—
			summer	4.7	6.7	8.3	9.7	10.4	11.6	12.9	13.9	—
LA	ST	4	daily	4.8	6.5	7.8	8.8	9.3	10.2	10.2	10.3	10.3?
			winter	4.8	6.7	7.6	8.8	9.5	10.3	10.3	10.3	10.3?
			summer	4.8	6.4	7.4	8.5	9.5	9.9	9.9	9.8	9.9?

cutoff frequency of 0.4 cycles per day is very common in the power spectra of any daily average data in the atmosphere. On the other hand, the power spectrum itself cannot distinguish chaotic signals from noisy periodic or quasi-periodic signals. Therefore, the computation of fractal dimensions or Lyapunov exponents becomes necessary to verify the existence of chaos and to extract the dynamics from the time series.

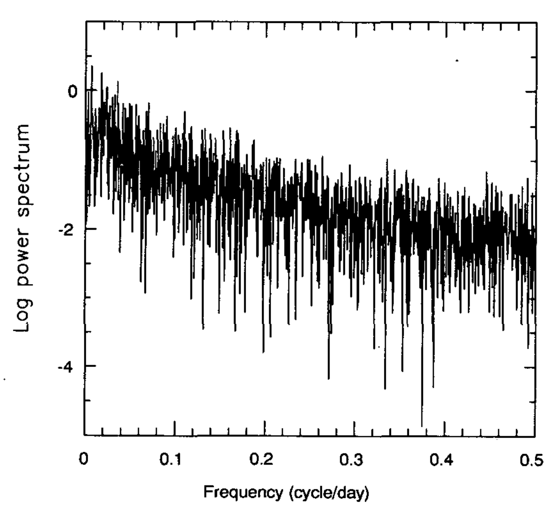

FIG. 2. The power spectrum of the daily surface temperature at Fort Collins, Colorado.

The fractal dimension is one of the commonly used measures of the "strangeness" of attractors. It is related to the number of degrees of freedom. It also provides statistical information about the system. Among the different procedures that have been developed to compute fractal dimensions are the nearest-neighbor method (Badii and Politi 1985), the correlation-integral method (Grassberger and Procaccia 1983a), and the singular-system method (Broomhead and King 1986). Some information about the quality of the results obtained with the different methods has been reported (Holzfuss and Mayer-Kress 1986). In practice, the correlation-integral method is the most widely used and is the one applied in this paper. The correlation dimension ν_s given by Grassberger and Procaccia (1983a) provides a rigorous lower bound to the information dimension and the Hausdorff dimension, and all three are generally close in value. The correlation dimension also provides a lower bound to the number of independent variables necessary to describe the time evolution of the dynamical system, and the Whitney embedding theorem (Takens 1981) gives an upper bound of $(2\nu_s + 1)$ to model the dynamics of a simple system. However, for complex systems such as the atmosphere, the conditions of the Whitney embedding theorem may not be satisfied, and this upper bound may not be valid.

The attempts to compute fractal dimensions in the past have been motivated by the speculation that climatic fluctuation may be governed by low-dimensional attractors. It will be shown that, at least for surface observational daily data, no low-dimensional attractors exist. Even if the saturated fractal dimension ν_s can be obtained, it may still be impossible to estimate the sufficient number of independent variables needed to model the atmospheric dynamics, and it is impossible to construct simple equations to describe the dynamics. This does not necessarily mean that all calculations about fractals in the atmosphere are useless. Pierrehumbert (1990) argued that they may tell us something about the statistics of atmospheric variability. The dynamical aspects of the fractality have also been considered for 2D turbulence by Osborne and Caponio (1990).

In a k-dimensional phase space [cf. Eq. (1)], the correlation function is given by (Grassberger and Procaccia 1983a)

$$C(r) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} H(r - \|\mathbf{x}_i - \mathbf{x}_j\|), \qquad (2)$$

where N is the total number of data points, H is the Heaviside function defined by $H(y) = 1$ for positive y and $H(y) = 0$ otherwise, and the usual Euclidean norm is used:

$$\|\mathbf{x}_i - \mathbf{x}_j\| = [\sum_{l=0}^{k-1} (x_{i+lm} - x_{j+lm})^2]^{1/2}. \qquad (3)$$

When the value of r is much smaller than the horizontal extent of the data, but larger than scales where measurement errors or noise are important, it can be shown that $C(r)$ depends upon r as (Grassberger and Procaccia 1983a)

$$C(r) \sim r^{\nu}. \qquad (4)$$

For each embedding dimension k, this exponent ν can be obtained from the slope of the linear part of a plot of $\ln C(r)$ versus $\ln r$. If ν approaches a value independent of k as $k \to \infty$ (usually $k > 2\nu$ is sufficient), this value is defined as the correlation dimension ν_s.

Figure 3 shows the plot of $\ln C(r)$ versus $\ln r$ for embedding dimensions $k = 5, 7, \cdots, 19$ for the sea surface pressure at BOX 139. Figure 4 shows the dimensionality ν of the weather attractor as a function of the number k of phase-space coordinates for the same time series. It is seen from Fig. 3 that, when r is very small, there are insufficient statistics, and the influence of noise inherent in the system or contributed by measurements is important; whereas, for r too large, the information is affected by nonlinearity, and the slope of the curve is smaller than that for intermediate

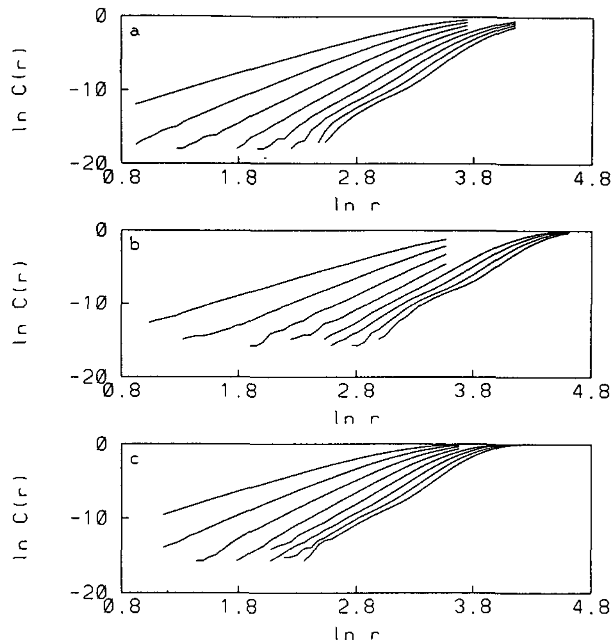

FIG. 3. Plots of $\ln C(r)$ versus $\ln r$ for embedding dimensions $k = 5, 7, \cdots, 19$ (ordered from left to right) for the sea surface pressure at BOX 139. (a) Daily data; (b) winter data; (c) summer data.

r. However, an intermediate range of r exists in which the slope is almost constant and Eq. (4) can be used.

For this linear range of r, a relatively easy procedure to compute the Kolmogorov entropy K (see Eckmann and Ruelle 1985) was also developed by Grassberger and Procaccia (1983b). The cumulative distribution $C_k(r)$ obtained from Eq. (2), where the subscript k refers to the embedding dimension, may be interpreted as the probability of finding two pieces of the trajectory whose distance remains less than r during the evolution time $(k-1)\tau$. When the embedding dimension is increased from k to $k+1$ at fixed r, the change from $C_k(r)$ to $C_{k+1}(r)$ gives the number of pairs of such trajectories escaping from a ball of radius r. With this interpretation, it can be argued that

$$C_k(r) \sim r^{\nu} e^{-k\tau K}. \qquad (5)$$

When saturation is reached for sufficiently large k, Eq. (5) with fixed r can be used to obtain the Kolmogorov entropy K:

$$K = \frac{1}{n\tau} \ln \frac{C_k(r)}{C_{k+n}(r)}, \qquad (6)$$

where the value of r should be within the linear part of the plot of $\ln C_k(r)$ versus $\ln r$ (e.g., see Fig. 3).

It is widely accepted now that there are limitations to the Grassberger–Procaccia algorithm when the number of data is limited. Qualitatively, because of the limitations on the number of observations, the interval in r for which Eq. (4) is valid begins to shrink as the

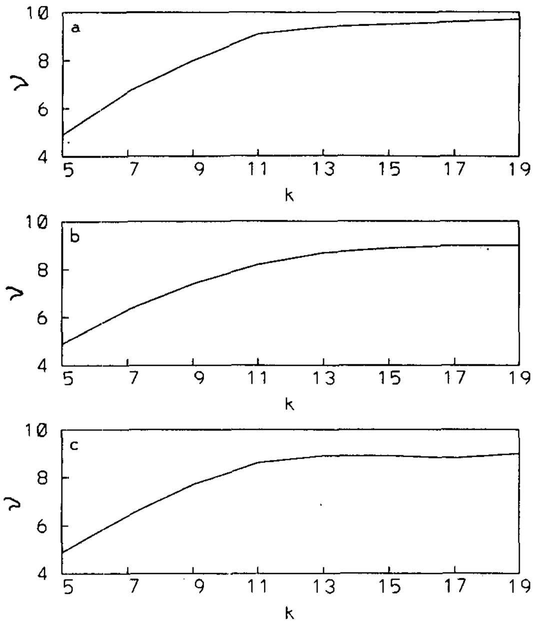

FIG. 4. Plots of the dimensionality ν as a function of embedding dimension k for the same circumstances as in Fig. 3.

embedding dimension k increases, and, when $k > 13$ in Fig. 3, this interval is very small, which means that estimates of ν for $k > 13$ are not so reliable. Similar arguments have been emphasized by Essex et al. (1987) and Tsonis and Elsner (1990) using the concept of critical embedding dimension k_c; that is, the embedding dimension above which the scaling region cannot be accurately defined. Quantitatively, according to Ruelle (1990), at least $M = 10^{\nu_s/2}$ data points are necessary to reliably estimate fractal dimension ν_s. With this in mind, Ruelle (1990) has shown that the estimate of $\nu_s = 7.3$ in Tsonis and Elsner (1988) is spurious. Using the same argument, it can be shown that the estimate of $\nu_s = 7.0$ in Fraedrich (1987) is also an artifact of the short time series (only 1680 daily data of the surface pressure for 14 winter seasons at Berlin) and thus is unreliable. [By the way, Fraedrich (1986) did mention that 5475 daily data for 15 years only lead to unsaturated fractal dimensions.] The estimate of a fractal dimension of about 8.0 in Keppenne and Nicolis (1989) may also be unreliable, because only about 9000 daily 500-mb geopotential records were used for each station over western Europe.

The quantitative data requirement for the Grassberger–Procaccia algorithm has been a highly debated subject in recent years, and various researchers have given different criteria, with that of Ruelle (1990) being the least strict. Although the data in Essex et al. (1987) and Sharifi et al. (1990) satisfy the requirement of Ruelle (1990), their estimates of low-dimensional attractors are still not so reliable according to the estimate of Nerenberg and Essex (1990) with the condition that the critical embedding dimension k_c be greater than the fractal dimension ν_s. The requirement that $k_c > \nu_s$ seems necessary in order to guarantee the saturation of the fractal dimension. By mentioning these examples (more can easily be found in the recent literature), we illustrate that most, if not all, of the previous estimates of low-dimensional attractors in the atmosphere are unreliable. However, Tsonis et al. (1991) recently relaxed the requirement $k_c > \nu_s$, using $k_c = \nu_s$ instead, and claimed that the existence of low-dimensional attractors in weather and climate should not be disregarded based on Nerenberg and Essex (1990). Finally, based on simple models, Lorenz (1991) recently proposed that, if a low fractal dimension can be obtained from observational data, this may instead reflect the weak nonlinear interaction between the observed variable and the other variables in the atmosphere.

Table 1 summarizes the dependence of ν on k for all data. Though saturation values seem to be approached for the sea surface pressure at BOX 244 and BOX 139, or for the surface temperature at Los Angeles, they are unreliable based on the above qualitative and quantitative arguments. It is also found from Table 1 that no saturation values can be reached for the sea surface temperatures at BOX 139 or BOX 244 or for the surface pressure and temperature at Fort Collins. Therefore, saturated fractal dimensions cannot be obtained for all datasets in the paper. For the total length of this data, it can only be concluded that the saturated fractal dimension is well above 8.

Table 2 shows the values of K computed from Eq. (6) for the sea surface pressures at BOX 244 and BOX 139 and the surface temperature at Los Angeles, where spurious saturated fractal dimensions are reached. The error-doubling time T, also given in Table 2, is computed from

$$T = (\ln 2)/K. \tag{7}$$

Since saturated fractal dimensions are not actually reached, Eq. (6) should not be used, and the error-doubling times in Table 2 are unreliable. For example, although the potential predictability is larger in BOX 244 than that in BOX 139 (Madden 1976), the error-doubling times for both regions are similar in Table 2. Also, for the Lorenz system (Lorenz 1963), or other known systems, our computations show that the values

TABLE 2. Kolmogorov entropy K and error-doubling time T of the sea surface pressure (SSP) at BOX 139 and BOX 244 and of the surface temperature (ST) at Los Angeles (LA).

	Location Variable	BOX 244 SSP	BOX 139 SSP	LA ST
	daily	0.095	0.093	0.173
K(day^{-1})	winter	0.061	0.079	0.159
	summer	0.084	0.068	0.155
	daily	7.3	7.5	4.0
T(day)	winter	11.4	8.8	4.4
	summer	8.3	10.2	4.5

of K and T are not sensitive to the time delay τ, whereas the values of K and T in Table 2 vary significantly when the time delay τ is increased. For example, in contrast to the values in Table 2 (with τ as given in Table 1), T is 14.6 days for BOX 244 SSP with $\tau = 10$ days, T is 8.3 days for BOX 139 SSP with $\tau = 8$ days, and $T = 6.3$ days for LA ST with $\tau = 6$ days. Since saturated fractal dimensions were not reached in previous studies, their estimates of the error-doubling time based on Eqs. (6) and (7) are also unreliable (e.g., in Fraedrich 1987).

The dependence of ν on k has also been computed for random data produced by a random-number generator. It is found that the relationship between ν and k for the observational data is similar to that for random data of a similar length (number of observations). (Note: for random data of infinite length, $\nu = k$.) However, the above data are not random, as shown by their power spectra and autocorrelation functions (Note: for Gaussian white noise, the autocorrelation function is zero; i.e., random data are independent of each other). The above results only show that the correlation dimension ν_s is so large that even an embedding dimension of $k = 19$ is not sufficient; that is, the dynamics of the weather attractors are controlled by too many degrees of freedom, and no low-dimensional attractors exist.

Caputo et al. (1986) pointed out that saturation could be reached to obtain spurious correlation dimensions even in very high dimensional embeddings for any dynamical system whatsoever, including cases of infinite-dimensional, stochastic signals. Recently, Osborne and Provenzale (1989) proved that a simple class of colored random noises showing a power-law decay have a finite and predictable value for the correlation dimension. Therefore, it is necessary to verify that the observed data are indeed deterministic chaos rather than colored noise. First, the autocorrelation function approaches zero slowly for all data (e.g., see Fig. 1c). Second, it is impossible to fit a power law to the spectrum over the whole range of frequencies (e.g., see Fig. 2). Finally, the exponent of the power-law fit to the spectrum in Fig. 2 is about 1.7, which would indicate a fractal dimension of only about 2.9 for colored noise (Osborne and Provenzale 1989). This value is much smaller than those in Table 1. Therefore, we can conclude that the data are not colored noise.

Only single-point time series are used in our paper and in most previous publications. In contrast, Pierrehumbert (1990) used the entire datasets of the monthly average heights of the 500-mb surface covering the Northern Hemisphere poleward of 30°N latitude and showed that these datasets are also insufficient to draw a firm conclusion about the dimensionality of the global atmosphere and that no low-dimensional (<20) global attractors exist.

The global attractor is probably multifractal (Halsey et al. 1986). This means that the globally averaged dimension will be dominated by the larger local dimensions; many local dimensions may be smaller than the global dimension, and the dimension may vary with location. Corresponding to this, the unsaturated dimensions $\nu(k)$ in Table 1 vary qualitatively with location. The geographic variability of fractal dimensions may also be related to the climatic signal-to-noise ratio. For instance, $\nu(k)$ for the surface temperature in Los Angeles is lower than in Fort Collins where the climatic signal-to-noise ratio is smaller (Madden and Shea 1978). On the other hand, the climatic signal-to-noise ratio in BOX 244 is higher than that in BOX 139 (cf. Fig. 6 in Madden 1976), but the values of $\nu(k)$ for the sea surface pressure in both regions are similar. However, these speculations about the geographical variability of fractal dimensions need further verification, since no saturated fractal dimensions can be obtained with confidence in Table 1. Similarly, although the correlation dimension of daily data appears to be larger than those of winter and summer data in Table 1, due to the fact that the (total) daily data include the transition-season (spring and fall) data as well, this also needs further verification.

In summary, this section has shown that most, if not all, of the previous estimates of low-dimensional attractors in the atmosphere are unreliable. Using longer time series of observational data, we still cannot obtain saturated fractal dimension ν_s and can only claim that ν_s is well above 8. Because saturated values cannot be reached, the computation of the Kolmogorov entropy and the error-doubling time based on Eqs. (6) and (7) is unreliable. It has also been shown that our data are neither white noise nor colored noise. The geographic variability of fractals is qualitatively discussed in relation to the global multifractal assumption and the climatic signal-to-noise ratio.

5. Lyapunov exponents and predictability of the atmosphere

In section 4 it is demonstrated that no saturated fractal dimensions can be reached and predictability cannot be estimated reliably. Though some people may question the usefulness of computing fractal dimensions for the atmosphere, nobody questions the importance of the predictability problem. Therefore, in this section, the Kolmogorov entropy and the error-doubling time in the atmosphere are evaluated by estimating the Lyapunov-exponent spectrum from a practical and reliable method (Zeng et al. 1991).

The Lyapunov exponent (which is sometimes called the Lyapunov number) is a quantitative measure of the sensitivity to initial conditions (i.e., the divergence of neighboring trajectories exponentially in time). The Kolmogorov entropy K is the sum of all positive Lyapunov exponents. The number of Lyapunov exponents is the same as the dimension of phase space. One of the Lyapunov exponents is necessarily equal to zero, meaning that the change of the relative distances of initially close states is slower than exponential, and the sum of all exponents must be strictly negative in a well-behaved dissipative system (Guckenheimer and Holmes 1983). In addition to a noninteger dimension for the attractor, positive Lyapunov exponents and positive, finite Kolmogorov entropy are basic quantities characterizing chaotic behavior. All three quantities are invariant under a smooth transformation of coordinates. There are several relations among these quantities, and, if the Lyapunov spectrum can be obtained for a given system, the other two quantities can be estimated or bounded (Fredrickson et al. 1983). However, for a very complex system (e.g., the atmosphere), all of the Lyapunov exponents cannot be determined accurately, so that these relations (e.g., the Kaplan–Yorke conjecture) cannot be used.

Though the Lyapunov-exponent spectrum can be estimated relatively easily for known systems [e.g., the Lorenz equations, Lorenz (1963)], it is difficult to obtain them from a time series of a complex system (e.g., the atmosphere). Wolf et al. (1985) proposed a method to estimate one or two positive Lyapunov exponents. Sano and Sawada (1985) developed a procedure to determine several Lyapunov exponents (including positive, zero, and even negative ones). A similar method was developed independently by Eckmann et al. (1986). Recently, Bryant et al. (1990) and Briggs (1990) also proposed to evaluate Lyapunov exponents by means of polynomial approximations, rather than the linear approximations used in Sano and Sawada (1985) and Eckmann et al. (1986). However, these methods require a large amount of data and/or fairly high precision.

Motivated by the work of Sano and Sawada (1985) and Eckmann et al. (1986), we have developed a similar algorithm to evaluate the Lyapunov-exponent spectrum (Zeng et al. 1991). Our scheme has been tested on various known systems that are finite or infinite dimensional, and this has shown that reasonable Lyapunov-exponent spectra can be obtained from only 5000 or 10 000 data points with a precision of 10^{-1} or 10^{-2} in a phase space whose dimension is less than six. Zeng et al. (1992) gives an extensive discussion of this and other methods. The basic steps of our method are as follows:

(i) The dynamical system is projected onto a k-dimensional phase space reconstructed by the time-delay method [cf. Eq. (1)].

(ii) A linear operator is determined to describe the evolution during the time $n\Delta t$ of small vectors $(\mathbf{x}_j - \mathbf{x}_i)$ originating at the point $\mathbf{x}_i$ to small vectors $(\mathbf{x}_{j+n} - \mathbf{x}_{i+n})$ originating at $\mathbf{x}_{i+n}$. If these vectors are sufficiently small that they can be regarded as good approximations to tangent vectors in the tangent space of the dynamical system, a $k \times k$ matrix T_i can be obtained from

$$\mathbf{x}_{j+n} - \mathbf{x}_{i+n} = T_i(\mathbf{x}_j - \mathbf{x}_i). \quad (8)$$

The least-square error algorithm can be used to obtain the elements of the matrix T_i, which is relatively simple when $n = m$ (i.e., $n\Delta t = \tau$).

(iii) The Q_iR_i decomposition of the matrix T_i, where Q_i is an orthogonal matrix and R_i is an upper-triangular matrix with nonnegative diagonal elements, is computed. The Lyapunov exponents λ_l are then obtained by averaging the logarithms of the diagonal elements of the matrix R_i (for more details see Zeng et al. 1991). In this paper, we take $n = m$ (i.e., $\tau = n\Delta t$) in Eq. (8). The number of matrices T_i is 1500, the number of $\mathbf{x}_j$ for each $\mathbf{x}_i$ is 10 to 12, which is sufficient to use the least-squares algorithm, and the embedding dimension k is varied from 2 to 9.

Noise is an infinite-dimensional process and tends to decrease the density of data points defining the attractor as the embedding dimension k increases (Wolf et al. 1985). Because of the limitations due to the noise level and the total number of data points, we believe that the results with $k \leqslant 5$ are more reliable than those with $k > 5$. Since our goal is to obtain with confidence as many Lyapunov exponents as possible, results for $k = 5$ only are reported. As shown in Zeng et al. (1991), at least the positive Lyapunov exponents can be computed reliably for $k = 5$. (In our computations, we also find that, as k increases beyond 5, the Lyapunov exponents decrease slowly, but the Kolmogorov entropy and the error-doubling time change very little.)

Table 3 summarizes the Lyapunov-exponent spectrum for each set of data analyzed in this paper. It is seen from Table 3 that at least two Lyapunov exponents are positive with comparable magnitude. Furthermore, at least one exponent must be zero, and this exponent is easily identified as λ_3 in each case (since λ_3 is zero to within the error bars). Therefore, the atmosphere has a hyperchaotic attractor with a folded, multidimensional fractal structure, and unstable motion of comparable importance occurs along two directions. We then obtain the Kolmogorov entropy, K, as the sum of the two positive Lyapunov exponents and the error-doubling time T for each dataset. It can be concluded from Table 3 that the predictability time T is about 5 to 8 days at BOX 139 and BOX 244, about 4 to 5 days in Los Angeles, and about 2 to 3 days in Fort Collins. It is also seen from Table 3 that the predictability time T is larger in summer than in winter for all variables, and T is shorter for the daily data than for the summer and/or winter data, since the daily data includes not only summer and winter data, but also transition-season (spring and fall) data.

Local predictability is controlled by large-scale advection processes and local forcing. Due to the regulation of the weather and climate in Los Angeles by the eastern portion of the subtropical ridge associated with the descending portion of the Hadley cell and the much more frequent influence of the polar front on the weather and climate in Fort Collins, the potential climatic predictability, which is a sort of climatic signal-to-noise ratio, is higher in Los Angeles than in Fort Collins (e.g., see Madden and Shea 1978). Corresponding to this, as shown in Table 3, the predictability

TABLE 3. Lyapunov-exponent spectrum with the parameters given in the text. The error-doubling time T is computed from Eq. (7), where the Kolmogorov entropy K is obtained by summing the first two Lyapunov exponents.

Location	Variable	Number of daily data	Data	Lyapunov exponent (day^{-1})					T (day)
			daily	0.063	0.031	−0.006	−0.052	−0.134	7.4
BOX 139	SST	13 870	winter	0.064	0.026	−0.005	−0.047	−0.131	7.7
			summer	0.064	0.019	−0.011	−0.059	−0.142	8.4
			daily	0.098	0.044	0.004	−0.047	−0.133	4.9
BOX 139	SSP	13 855	winter	0.117	0.055	0.021	−0.048	−0.142	4.0
			summer	0.075	0.037	−0.000	−0.042	−0.136	6.2
			daily	0.102	0.046	0.001	−0.042	−0.132	4.7
BOX 244	SST	13 860	winter	0.086	0.040	0.006	−0.048	−0.130	5.5
			summer	0.078	0.030	−0.007	−0.043	−0.138	6.4
			daily	0.068	0.033	0.003	−0.031	−0.089	6.9
BOX 244	SSP	13 877	winter	0.078	0.034	0.006	−0.028	−0.098	6.2
			summer	0.060	0.028	−0.002	−0.033	−0.096	7.9
			daily	0.195	0.081	0.016	−0.077	−0.220	2.5
FCL	ST	36 555	winter	0.191	0.089	0.011	−0.063	−0.228	2.5
			summer	0.144	0.058	−0.010	−0.068	−0.228	3.4
			daily	0.283	0.119	0.022	−0.089	−0.348	1.7
FCL	SP	32 870	winter	0.238	0.089	−0.000	−0.108	−0.340	2.2
			summer	0.215	0.105	0.000	−0.122	−0.325	2.2
			daily	0.121	0.065	0.004	−0.059	−0.174	3.7
LA	ST	14 245	winter	0.122	0.053	−0.001	−0.058	−0.169	4.0
			summer	0.101	0.046	−0.002	−0.052	−0.162	4.7

time T is larger for surface temperature in Los Angeles than in Fort Collins. It is also larger for sea surface pressure in BOX 244 than in BOX 139 where the climatic signal-to-noise ratio is also smaller and the potential predictability is shorter (Madden 1976).

Our estimates of the error-doubling time in Table 3 are from about 2 to 8 days for various locations. However, twin experiments of general circulation models (GCMs) by Smagorinsky (1969) give an error-doubling time of about 2.5 days for the vertically integrated standard deviation of the temperature in the Northern Hemisphere. Using the ECMWF GCM, Lorenz (1982) obtained an error-doubling time of about 2 days for small initial errors of 500-mb heights in the Northern Hemisphere with the aid of a quadratic hypothesis for the nonlinear terms in the equation governing the growth of errors. The difference between their estimates and our estimates may be explained as follows. First, we use single-point time series and study local predictability, but they used data for the entire Northern Hemisphere and studied the global predictability, which is controlled by areas of lower local predictability (i.e., higher Kolmogorov entropy). Therefore, it is natural that our estimates of the error-doubling time are approximately equal to or greater than their estimates. Second, the error-doubling time depends on the magnitude of initial errors: this time is short for small initial errors and long for larger initial errors. Because our data is from observations, the estimates of error-doubling time are generally larger (and more physically relevant) than those based on truly small initial errors. Third, daily averaged data are used in this paper, but instantaneous values were used in their studies. The averaging process smooths data and increases the error-doubling time. Fourth, we use observational data, but they used model-generated data; it is not clear that the error-doubling time in the GCMs is the same as in the real atmosphere. Finally, we use surface observations, but they used results above ground. The last two points will cause differences between our estimates of error-doubling time and theirs, but do not necessarily increase the error-doubling time in our estimates. Using the 500-mb geopotential record over western Europe, Keppenne and Nicolis (1989) obtained a error-doubling time of about 19 days. This discrepancy with our estimates may also be explained by the second and last points in above discussions.

Finally, a problem that needs further study using more data is discussed. For known systems (e.g., the Lorenz equations), we found that the computation of the Lyapunov-exponent spectrum is insensitive to the selection of τ, which is approximately the e-folding time of the autocorrelation function. For Gaussian white noise, our computations show that Lyapunov exponents are inversely proportional to τ. However, when our data were analyzed, it was found that the Lyapunov exponents decrease slightly as τ increases, although they are not inversely proportional to τ, as would be the case for Gaussian white noise. This may be due to the high fractal dimension of the atmosphere. This uncertainty could increase the estimates of T by as much as 50%. Much more data may be required so that the

Lyapunov-exponent spectrum can be evaluated for higher embedding dimensions to improve the estimates of the predictability time T.

Since the noise level in our data may be assumed to be typical for daily data of the atmosphere, and since the total number of data points (from about 13 800 to about 36 500) is larger than those used in previous studies [e.g., about 5500 in Fraedrich (1986); about 9000 for each station in Keppenne and Nicolis (1989)], it is not unreasonable to assume that the above problem may be very common for similar studies using daily data of the atmosphere, and sensitivity studies should be conducted to determine its importance in estimating predictability.

6. Discussion

We have applied theories of deterministic chaos and fractal structure in dissipative dynamical systems to the computation of dimensionality and predictability from time series of daily data over various regions of the United States and the North Atlantic Ocean. Because of limitations to the Grassberger–Procaccia algorithm for limited numbers of data, we have shown that most, if not all, of the previous estimates of low-dimensional attractors in the atmosphere are unreliable. Although the noise level of our datasets may be assumed to be typical for daily data of the atmosphere, and the total number of data points is larger than in previous work, we still cannot obtain saturated values for the correlation dimensions from the sea surface temperature or pressure in the Atlantic or from the surface pressure or temperature at Fort Collins, Colorado, or Los Angeles, California. If the saturation values could be reached, they would be well over 8. Based on these results, we may claim that probably no low-dimensional strange attractors exist. Because ν_s cannot be reached, the calculation of the Kolmogorov entropy and the error-doubling time based on the computation of dimensions [Eqs. (6) and (7)] is also unreliable.

In contrast to these unreliable estimates of dimensions and error-doubling times of atmospheric data from the Grassberger–Procaccia algorithm, we present a practical method to evaluate the Kolmogorov entropy and the error-doubling time by computing the Lyapunov-exponent spectrum. In this way, we obtain error-doubling times varying from about 2 to 3 days in Fort Collins to about 4 to 5 days in Los Angeles and to about 5 to 8 days in BOX 139 (midlatitude North Atlantic) and BOX 244 (subtropical North Atlantic). These time scales are smaller than those inferred by Keppenne and Nicolis (1989) for western Europe and are approximately equal to, or larger than, those obtained by Smagorinsky (1969) and Lorenz (1982) using GCMs. The reasons are discussed in detail. The predictability time in an area (e.g., Los Angeles) of high climatic signal-to-noise ratio is longer than that in an area (e.g., Fort Collins) where the climatic signal-to-noise ratio is small.

The fractal dimensions, Lyapunov exponents, and predictability are also discussed for summer and winter seasons. Since the daily data includes not only summer and winter data, but also data for the transition seasons (spring and fall), whose predictability is poorer and whose dynamics are controlled by more variables than in summer and winter, it is expected that the predictability time for daily data should be shorter than for summer and/or winter data, which is supported by our results. It is also expected that the fractal dimensions of daily data should be higher than those of summer and/or winter data, which is also suggested by our results but needs further verification.

Traditional predictability studies by means of numerical models in atmospheric science usually study the error growth from a reference state disturbed by various methods of perturbation (ECMWF Workshop 1988). Both the traditional approach and the analysis presented in this paper share a common conclusion: when the initial error is small, its growth rate depends on the dynamics of the system, rather than on the initial error itself. On the other hand, traditional predictability studies provide only the largest (positive) Lyapunov exponent, whereas our analysis provides all of the positive exponents, which allows a more appropriate quantitative measure of (un)predictability since different positive Lyapunov exponents correspond to the divergence of initial errors in different directions. The traditional approach uses entire global fields of data; in contrast, we only use single-point time series. We do not expect to extract all of the information about the global dynamics, but we do expect to gain an insight from local time-series analyses. We also anticipate that this kind of analysis will aid the selection of initial states for the Monte Carlo ensemble extended-range forecast.

Finally, we have shown that the calculations of the Lyapunov-exponent spectrum, the Kolmogorov entropy, and the error-doubling time are slightly sensitive to the choice of the time delay τ. These sensitivities may be caused by the high fractal dimension in the atmosphere. Since such sensitivity was not mentioned in similar previous studies, further research with more data is needed to clarify this problem.

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