The Derivation of a Terrain-Following Coordinate System for Use in a Hydrostatic Model

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ABSTRACT

This article uses tensor transformation procedures in order to derive a terrain-following coordinate system that is frequently used in a number of regional and mesoscale hydrostatic models. Tensor transformation procedures are used so as to ensure physical invariance of the primitive equations between the Cartesian and terrain-following systems. Among the major conclusions are as follows:

- Applying the chain rule to the hydrostatic equation, before transforming from a Cartesian to a terrain-following coordinate system, yields a different set of equations than if the hydrostatic assumption is applied *after* the tensor transformation is made. The hydrostatic equations in the two terrain-following representations are the same *only when* the slope of the terrain in the model is much less than 45°.
- Variations of the metric tensor across a grid volume appear in the set of conservation equations as a result of averaging the equations over a grid volume. Such deviations have always been ignored in existing non-hydrostatic and hydrostatic meteorological models.
- Care must be taken to assure that parameterizations which are a function of distance above the ground be defined in terms of the original Cartesian system, and not the new generalized vertical coordinate σ. The profile exchange coefficient $K(z)$, for example, cannot be defined simply by replacing z by σ.

1. Introduction

The use of a terrain-following coordinate system in meteorological modeling was first introduced by Phillips (1957), and it has since been shown to be an effective mathematical representation. This concept of defining a coordinate surface coincident with the bottom topography permits more efficient use of computer resources, and it simplifies the application of lower boundary conditions. In Phillips' original form, adopted by many models [e.g., the U.S. Weather Service forecast models, Rieck (1979)], pressure is used to define the independent vertical coordinate σ, where surface pressure is used as the lower boundary. Haltiner (1971), for example, defines $\sigma = p/p_s$, where p_s is the surface pressure while p is the pressure at any level. For this example $\sigma = 1$ corresponds to the ground surface.

In recent years, σ has often been defined as a function of height rather than pressure. This is advantageous because p_s is a function of time, whereas terrain height is not. The general form of this coordinate system transformation is given as

$$\sigma = s\frac{z - z_G}{s - z_G}\,, \tag{1}$$

where s is usually defined as a constant (generally defined as the top of the model)[1] while z_G is the terrain height. The variable z is height, while σ is referred to as a generalized vertical coordinate. This form of a terrain-following coordinate has been used in recent years in regional and mesoscale models (e.g., Mahrer and Pielke, 1975; Kasahara, 1974; and others) in which the hydrostatic assumption has been applied.

In developing their equations, however, these investigators have applied the chain rule *separately* in the vertical and horizontal dimensions (utilizing the hydrostatic relation). Using (1), this results in the transformed hydrostatic equation given as

$$\frac{\partial\pi}{\partial\sigma} = -\frac{s - z_G}{s}\frac{g}{\theta}\,, \tag{2}$$

where $\pi = c_pT/\theta$. This is appropriate if the hydrostatic assumption is *exactly* satisfied. However, the invariance of the physical representation (which must be retained, regardless of the coordinate formulation) is lost if the assumption is not exact,

[1] Mahrer and Pielke (1975, 1978) permitted s to vary in the denominator, but this complicates the analyses somewhat. Therefore, in this paper, s is assumed to be a constant. The conclusions from the analyses are not significantly affected by this requirement. See Dutton (1976, p. 251) for the modifications needed when the coordinate transformation is time dependent.

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as discussed by Dutton (1976, p. 252). On the synoptic scale, in which horizontal scales are always much larger than the vertical scales of motion, this requirement is very closely satisfied.

On the mesoscale, however, it may be more appropriate to perform a tensor transformation of all three components of the equation of motion, before making the hydrostatic assumption. As shown by Gal-Chen and Somerville (1975), and Clark (1977), for use in a non-hydrostatic model, a rigorous transformation between coordinate systems requires use of the properties of tensor analysis in order to assure that the invariance of the physical representation is retained in all coordinate systems.

In this paper, we perform a tensor transformation of the equations of motion and then apply the hydrostatic assumption. The resulting equations reduce to the form used by Mahrer and Pielke (1975), Kasahara (1974) and others only when certain simplifying assumptions are made. Moreover, since the hydrostatic assumption is applied later in the derivation of the transformed equations, a more in-depth understanding of the coordinate transformation is obtained.

2. The equation of motion[2]

Dutton (1976) demonstrated that the contravariant form of the equation of motion in a generalized coordinate system, derived from the rectangular x-y-z (x^i) system, can be written as

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^j \tilde{u}^i_{;j} = -\tilde{G}^{ij}\theta \frac{\partial \pi}{\partial \tilde{x}^j} - \frac{\partial \tilde{x}^i}{\partial z} g - 2\tilde{\epsilon}^{ijl}\tilde{\Omega}_j \tilde{u}_l, \quad (3)$$

where $\tilde{u}^i$ is the contravariant component of velocity, $\tilde{G}^{ij}$ the contravariant metric tensor, $\tilde{x}^j$ represents the independent variable in the new coordinate system, and

$$\tilde{\epsilon}^{ijl} = \epsilon_{ijl}\tilde{G}^{-1/2}$$

$$\tilde{u}^i_{;j} = \frac{\partial \tilde{u}^i}{\partial \tilde{x}^j} + \tilde{\Gamma}^i_{jl}\tilde{u}^l.$$

The term $g^{(\partial \tilde{x}^i/\partial z)}$ is obtained from $\tilde{G}^{ij}\partial\Phi/\partial\tilde{x}^j$. The tilde is used to indicate a variable in the transformed coordinate system, while $\epsilon_{ijl} = \epsilon^{ijl}$ in the Cartesian system. The tensor ϵ_{ijl} is defined as zero if any two of the indices are equal, +1 if an even permutation of the indices occur, and −1 with an odd permutation. The parameter $\tilde{G}$ is the determinant of the contravariant form of the metric tensor $\tilde{G}^{ij}$ while $\tilde{\Gamma}^i_{jl}$, called the Christoffel symbol, is given by

$$\tilde{\Gamma}^i_{jl} = \frac{\partial \tilde{x}^i}{\partial x^m}\frac{\partial^2 x^m}{\partial \tilde{x}^j \partial \tilde{x}^l}.$$

[2] The material in this section closely follows Dutton's (1976, pp. 248–250) discussion on generalized coordinates.

Eq. (3) is somewhat cumbersome to work with. However, it is essential to retain all the terms which arise from the transformation in order to preserve tensor invariance.

When applying these equations to simulate meteorological systems, only the vertical coordinate in the rectangular system is customarily transformed. In addition, it is necessary to average the transformed equations since (3) is only valid over spatial and temporal intervals which are much smaller than the space and time scales used in meteorological numerical models.

The functional form of this generalized vertical coordinate transformation, in terms of the original Cartesian system, can be written as

$$\tilde{x}^1 = x, \qquad x = \tilde{x}^1$$
$$\tilde{x}^2 = y, \qquad y = \tilde{x}^2$$
$$\tilde{x}^3 = \sigma(x, y, z, t), \qquad z = h(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3, t)$$

where σ can be given by (1), or another functional form such as discussed by Haltiner (1971) and Kasahara (1974).

The contravariant and covariant forms of the metric tensor $\tilde{G}^{ij}$ and $\tilde{G}_{ij}$, are given as

$$\tilde{G}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^l}\frac{\partial \tilde{x}^j}{\partial x^l}$$

$$= \begin{bmatrix} 1 & 0 & \dfrac{\partial\sigma}{\partial x} \\ 0 & 1 & \dfrac{\partial\sigma}{\partial y} \\ \dfrac{\partial\sigma}{\partial x} & \dfrac{\partial\sigma}{\partial y} & \left(\dfrac{\partial\sigma}{\partial x}\right)^2 + \left(\dfrac{\partial\sigma}{\partial y}\right)^2 + \left(\dfrac{\partial\sigma}{\partial z}\right)^2 \end{bmatrix},$$

$$\tilde{G}_{ij} = \frac{\partial x^l}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}$$

$$= \begin{bmatrix} 1 + \left(\dfrac{\partial h}{\partial \tilde{x}^1}\right)^2 & \dfrac{\partial h}{\partial \tilde{x}^1}\dfrac{\partial h}{\partial \tilde{x}^2} & \dfrac{\partial h}{\partial \tilde{x}^1}\dfrac{\partial h}{\partial \tilde{x}^3} \\ \dfrac{\partial h}{\partial \tilde{x}^1}\dfrac{\partial h}{\partial \tilde{x}^2} & 1 + \left(\dfrac{\partial h}{\partial \tilde{x}^2}\right)^2 & \dfrac{\partial h}{\partial \tilde{x}^2}\dfrac{\partial h}{\partial \tilde{x}^3} \\ \dfrac{\partial h}{\partial \tilde{x}^1}\dfrac{\partial h}{\partial \tilde{x}^3} & \dfrac{\partial h}{\partial \tilde{x}^2}\dfrac{\partial h}{\partial \tilde{x}^3} & \left(\dfrac{\partial h}{\partial \tilde{x}^3}\right)^2 \end{bmatrix},$$

while the only nonzero Christoffel symbol is

$$\tilde{\Gamma}^3_{jl} = \frac{\partial\sigma}{\partial z}\frac{\partial^2 h}{\partial \tilde{x}^j \partial \tilde{x}^l},$$

so that the covariant derivative of velocity is given by

$$u^i_{;j} = \begin{cases} \dfrac{\partial \tilde{u}^i}{\partial \tilde{x}^j}, & i = 1, 2 \\ \dfrac{\partial \tilde{u}^3}{\partial \tilde{x}^j} + \tilde{\Gamma}^3_{jl}\tilde{u}^l, & i = 3. \end{cases}$$

The determinant of the Jacobian of the transformation,

$$\left|\frac{\partial x^i}{\partial \tilde{x}^j}\right| = \tilde{G}^{1/2},$$

is given by

$$\left|\frac{\partial x^i}{\partial \tilde{x}^j}\right| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{\partial h}{\partial \tilde{x}^1} & \dfrac{\partial h}{\partial \tilde{x}^2} & \dfrac{\partial h}{\partial \tilde{x}^3} \end{vmatrix} = \tilde{G}^{1/2} = \frac{\partial h}{\partial \tilde{x}^3} \equiv \frac{\partial h}{\partial \sigma}.$$

The tangent and normal basis vectors for the generalized vertical coordinate system in terms of the rectangular representation are given by

$$\left.\begin{aligned} \boldsymbol{\tau}_1 &= \mathbf{i} + \mathbf{k}\frac{\partial h}{\partial \tilde{x}^1}, & \boldsymbol{\eta}^1 &= \mathbf{i} \\ \boldsymbol{\tau}_2 &= \mathbf{j} + \mathbf{k}\frac{\partial h}{\partial \tilde{x}^2}, & \boldsymbol{\eta}^2 &= \mathbf{j} \\ \boldsymbol{\tau}_3 &= \mathbf{k}\frac{\partial h}{\partial \tilde{x}^3}, & \boldsymbol{\eta}^3 &= \mathbf{i}\frac{\partial \sigma}{\partial x} + \mathbf{j}\frac{\partial \sigma}{\partial y} + \mathbf{k}\frac{\partial \sigma}{\partial z} \end{aligned}\right\} \quad (4)$$

where, since $\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$ does not equal zero when $i \neq j$, this coordinate system in general is *nonorthogonal*. In the original rectangular coordinate system, the normal and tangent basis functions are the same (i.e., **i**, **j** and **k**) and are orthogonal to one another.

The individual contravariant and covariant velocity components are found from $\tilde{u}^i = \boldsymbol{\eta}^i \cdot \mathbf{u}$ and $\tilde{u}_i = \boldsymbol{\tau}_i \cdot \mathbf{u}$, respectively, where $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, so that

$$\tilde{u}^1 = u, \qquad \tilde{u}_1 = u + \frac{\partial h}{\partial \tilde{x}^1} w$$

$$\tilde{u}^2 = v, \qquad \tilde{u}_2 = v + \frac{\partial h}{\partial \tilde{x}^2} w$$

$$\tilde{u}^3 = u\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial y} + w\frac{\partial \sigma}{\partial z}, \qquad \tilde{u}_3 = w\frac{\partial h}{\partial \tilde{x}^3}.$$

Kinetic energy is computed from these expressions by

$$e = \tfrac{1}{2}(\tilde{u}^1\tilde{u}_1 + \tilde{u}^2\tilde{u}_2 + \tilde{u}^3\tilde{u}_3).$$

As mentioned earlier, averaging of (3) is required if these equations are to be used in meteorological numerical models with finite grid and time intervals. The correct form of averaging this equation is called the grid-volume average and is given by

$$\overline{(\ \ )} = \int_t^{t+\Delta t}\int_{\tilde{x}^1}^{\tilde{x}^1+\Delta\tilde{x}^1}\int_{\tilde{x}^2}^{\tilde{x}^2+\Delta\tilde{x}^2}\int_{\sigma}^{\sigma+\Delta\sigma} (\ \ )\,d\sigma d\tilde{x}^2 d\tilde{x}^1 dt/(\Delta\tilde{x}^1)(\Delta\tilde{x}^2)(\Delta\sigma)(\Delta t). \quad (5)$$

The dependent variables can be decomposed into an average and a subgrid-scale perturbation expressed as

$$\phi = \bar{\phi} + \phi'',$$

where ϕ'' is a deviation from the grid-volume average. The symbol ϕ represents any one of the dependent variables.

Using (5), Eq. (3) can be rewritten as

$$\frac{\partial \bar{\tilde{u}}^i}{\partial t} = -\bar{\tilde{u}}^j \bar{\tilde{u}}^i_{;j} - \overline{\tilde{u}^{j''}\tilde{u}^{i''}_{;j}} - \tilde{G}^{ij}\bar{\theta}\frac{\partial \bar{\pi}}{\partial \tilde{x}^j} - \overline{\frac{\partial \tilde{x}^i}{\partial z}} g - 2\tilde{\epsilon}^{ijl}\tilde{\Omega}_j\bar{\tilde{u}}_l. \quad (6)$$

In deriving this form it has been assumed that $\theta = \bar{\theta}(1 + \theta''\bar{\theta}^{-1}) \approx \bar{\theta}$ and that

$$\bar{\bar{\tilde{u}}}^i = \bar{\tilde{u}}^i;$$

$$\overline{\frac{\partial u_i}{\partial t}} = \frac{\partial \bar{u}_i}{\partial t}, \quad \text{etc., (therefore } \overline{\tilde{u}^{i''}} = 0, \text{ etc.).} \quad (7)$$

Assumption (7), when applied in a rectangular coordinate system, is called Reynold's averaging. To make this assumption in the transformed coordinate system, however, it is necessary *to require that changes of the metric tensor over the four-dimensional grid-volume* $\Delta\tilde{x}^1\Delta\tilde{x}^2\Delta\sigma\Delta t$ *are small*, since this tensor appears in (6). Expressed mathematically, this requirement can be written as

$$\overline{\tilde{G}^{ij}} = \int_t^{t+\Delta t}\int_{\tilde{x}^1}^{\tilde{x}+\Delta\tilde{x}^1}\int_{\tilde{x}^2}^{\tilde{x}^2+\Delta\tilde{x}^2}\int_{\sigma}^{\sigma+\Delta\sigma} \tilde{G}^{ij} d\sigma d\tilde{x}^2 d\tilde{x}^1 dt/(\Delta t)(\Delta\tilde{x}^1)(\Delta\tilde{x}^2)(\Delta\sigma) \approx \tilde{G}^{ij}.$$

This requirement has significant implications on the choice of the vertical generalized coordinate since *it must be selected such that variations of the gradient of the transformed coordinate within the grid volume are small* compared with variations in the averaged gradient between adjacent grid volumes.

The advection term in (6) is derived from

$$\overline{\tilde{u}^j\tilde{u}^i_{;j}} = \overline{\tilde{u}^j \frac{\partial \tilde{u}^i}{\partial \tilde{x}^j}} + \overline{\tilde{\Gamma}^i_{jl}\tilde{u}^j\tilde{u}^l} \approx \overline{\tilde{u}^j \frac{\partial \tilde{u}^i}{\partial \tilde{x}^j}} + \bar{\tilde{\Gamma}}^i_{jl}\overline{\tilde{u}^j\tilde{u}^l}$$

$$\approx \bar{\tilde{u}}^j \frac{\partial \bar{\tilde{u}}^i}{\partial \tilde{x}^j} + \overline{\tilde{u}^{j''} \frac{\partial \tilde{u}^{i''}}{\partial \tilde{x}^j}} + \bar{\tilde{\Gamma}}^i_{jl}[\bar{\tilde{u}}^j\bar{\tilde{u}}^l + \overline{\tilde{u}^{j''}\tilde{u}^{l''}}]$$

$$= \bar{\tilde{u}}^j\bar{\tilde{u}}^i_{;j} + \overline{\tilde{u}^{j''}\tilde{u}^{i''}_{;j}},$$

where the assumption that changes of the metric tensor and its derivatives are small permits the removal of the Christoffel symbol from the integrand [this assumption can also be written as

$$\tilde{\Gamma}^i_{jl} = \bar{\tilde{\Gamma}}^i_{jl} + \tilde{\Gamma}''^i_{jl} = \bar{\tilde{\Gamma}}^3_{jl} + \tilde{\Gamma}''^3_{jl} \cong \bar{\tilde{\Gamma}}^3_{jl}$$

where

$$|\tilde{\Gamma}''^3_{jl}| \ll |\bar{\tilde{\Gamma}}^3_{jl}|].$$

The Coriolis term can be expanded as

$$2\tilde{\epsilon}^{ijl}\tilde{\Omega}_j\bar{\tilde{u}}_l = 2\tilde{\Omega}_j\tilde{G}_{lm}\bar{\tilde{u}}^m\epsilon_{ijl}\tilde{G}^{-1/2}$$

$$= 2\epsilon_{ijl}\frac{\partial x^r}{\partial \tilde{x}^j}\Omega_r\tilde{G}_{lm}\bar{\tilde{u}}^m\frac{\partial \sigma}{\partial z}$$

[with $\Omega_r = (0, |\Omega| \cos\phi, |\Omega| \sin\phi) = (0, \hat{f}/2, f/2)$, where Ω is the rotation rate of the earth and ϕ the latitude]. In addition,

$$\frac{\partial h}{\partial \tilde{x}^3}\frac{\partial \sigma}{\partial x^3} = 1,$$

$$\frac{\partial \tilde{x}^1}{\partial z} = \frac{\partial \tilde{x}^2}{\partial z} = 0.$$

With the decomposition of the variables into resolvable and subgrid scale terms, Eq. (6) can therefore be written for the generalized vertical coordinate representation in component form as

$$\frac{\partial \bar{\tilde{u}}^1}{\partial t} = -\bar{\tilde{u}}^j\frac{\partial \bar{\tilde{u}}^1}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''}\frac{\partial \tilde{u}^{1''}}{\partial \tilde{x}^j}} - \bar{\theta}\frac{\partial \bar{\pi}}{\partial \tilde{x}^1} - \bar{\theta}\frac{\partial \sigma}{\partial x}\frac{\partial \bar{\pi}}{\partial \tilde{x}^3} + \left(\hat{f} + \frac{\partial h}{\partial \tilde{x}^2}f\right)\left(\frac{\partial h}{\partial \tilde{x}^1}\bar{\tilde{u}}^1 + \frac{\partial h}{\partial \tilde{x}^2}\bar{\tilde{u}}^2 + \frac{\partial h}{\partial \tilde{x}^3}\bar{\tilde{u}}^3\right)$$
$$- f\left\{\frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^1}\bar{\tilde{u}}^1 + \left[1 + \left(\frac{\partial h}{\partial \tilde{x}^2}\right)^2\right]\bar{\tilde{u}}^2 + \frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^3}\bar{\tilde{u}}^3\right\}, \quad (8)$$

$$\frac{\partial \bar{\tilde{u}}^2}{\partial t} = -\bar{\tilde{u}}^j\frac{\partial \bar{\tilde{u}}^2}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''}\frac{\partial \tilde{u}^{2''}}{\partial \tilde{x}^j}} - \bar{\theta}\frac{\partial \bar{\pi}}{\partial \tilde{x}^2} - \bar{\theta}\frac{\partial \sigma}{\partial y}\frac{\partial \bar{\pi}}{\partial \tilde{x}^3} - \frac{\partial h}{\partial \tilde{x}^1}f\left(\frac{\partial h}{\partial \tilde{x}^1}\bar{\tilde{u}}^1 + \frac{\partial h}{\partial \tilde{x}^2}\bar{\tilde{u}}^2 + \frac{\partial h}{\partial \tilde{x}^3}\bar{\tilde{u}}^3\right)$$
$$+ f\left\{\left[1 + \left(\frac{\partial h}{\partial \tilde{x}^1}\right)^2\right]\bar{\tilde{u}}^1 + \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^2}\bar{\tilde{u}}^2 + \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^3}\bar{\tilde{u}}^3\right\}, \quad (9)$$

$$\frac{\partial \bar{\tilde{u}}^3}{\partial t} = -\bar{\tilde{u}}^j\frac{\partial \bar{\tilde{u}}^3}{\partial \tilde{x}^j} - \overline{\tilde{u}^{j''}\frac{\partial \tilde{u}^{3''}}{\partial \tilde{x}^j}} - \bar{\tilde{\Gamma}}^3_{jl}\bar{\tilde{u}}^j\bar{\tilde{u}}^l - \bar{\tilde{\Gamma}}^3_{jl}\overline{\tilde{u}^{j''}\tilde{u}^{l''}} - \bar{\theta}\left\{\frac{\partial \sigma}{\partial x}\frac{\partial \bar{\pi}}{\partial \tilde{x}^1} + \frac{\partial \sigma}{\partial y}\frac{\partial \bar{\pi}}{\partial \tilde{x}^2} + \left[\left(\frac{\partial \sigma}{\partial x}\right)^2 + \left(\frac{\partial \sigma}{\partial y}\right)^2\right.\right.$$
$$\left.\left. + \left(\frac{\partial \sigma}{\partial z}\right)^2\right]\frac{\partial \bar{\pi}}{\partial \tilde{x}^3}\right\} - \left(\hat{f} + \frac{\partial h}{\partial \tilde{x}^2}f\right)\frac{\partial \sigma}{\partial z}\left\{\left[1 + \left(\frac{\partial h}{\partial \tilde{x}^1}\right)^2\right]\bar{\tilde{u}}^1 + \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^2}\bar{\tilde{u}}^2 + \frac{\partial h}{\partial \tilde{x}^1}\frac{\partial h}{\partial \tilde{x}^3}\bar{\tilde{u}}^3\right\}$$
$$+ \frac{\partial h}{\partial \tilde{x}^1}f\frac{\partial \sigma}{\partial z}\left\{\frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^1}\tilde{u}^1 + \left[1 + \left(\frac{\partial h}{\partial \tilde{x}^2}\right)^2\right]\tilde{u}^2 + \frac{\partial h}{\partial \tilde{x}^2}\frac{\partial h}{\partial \tilde{x}^3}\tilde{u}^3\right\} - \frac{\partial \sigma}{\partial z}g. \quad (10)$$

To illustrate the effect of utilizing the hydrostatic assumption in (8)–(10), it is convenient to use (1) as the generalized vertical coordinate. The relation between the spatial coordinates in the two representations is given by

$$\left.\begin{array}{ll} \tilde{x}^1 = x, & x = \tilde{x}^1 \\ \tilde{x}^2 = y, & y = \tilde{x}^2 \\ \tilde{x}^3 = \sigma = s[z - z_G(x, y)]/(s - z_G(x, y)], & \\ & z = h = \dfrac{\sigma}{s}[s - z_G(\tilde{x}^1, \tilde{x}^2)] + z_G(\tilde{x}^1, \tilde{x}^2) \end{array}\right\}, \quad (11)$$

so that the nonzero quantities needed to evaluate the Jacobian, metric tensor and Christoffel symbol are given as

$$\left.\begin{aligned}
&\frac{\partial\sigma}{\partial x}=\frac{\partial z_G}{\partial x}\left(\frac{\sigma-s}{s-z_G}\right), \quad \frac{\partial h}{\partial \bar{x}^1}=\frac{\partial z_G}{\partial \bar{x}^1}\left(\frac{s-\sigma}{s}\right)\\
&\frac{\partial\sigma}{\partial y}=\frac{\partial z_G}{\partial y}\left(\frac{\sigma-s}{s-z_G}\right), \quad \frac{\partial h}{\partial \bar{x}^2}=\frac{\partial z_G}{\partial \bar{x}^2}\left(\frac{s-\sigma}{s}\right)\\
&\frac{\partial\sigma}{\partial z}=\frac{s}{s-z_G}, \quad \frac{\partial h}{\partial \sigma}=\frac{s-z_G}{s}
\end{aligned}\right\}, \tag{12}$$

$$\bar{\Gamma}^3_{11}=\frac{s-\sigma}{s-z_G}\frac{\partial^2 z_G}{\partial \bar{x}^{1^2}}, \quad \bar{\Gamma}^3_{22}=\frac{s-\sigma}{s-z_G}\frac{\partial^2 z_G}{\partial \bar{x}^{2^2}},$$

$$\bar{\Gamma}^3_{21}=\frac{s-\sigma}{s-z_G}\frac{\partial^2 z_G}{\partial \bar{x}^1\partial \bar{x}^2}, \quad \bar{\Gamma}^3_{23}=-\frac{1}{s-z_G}\frac{\partial z_G}{\partial \bar{x}^2},$$

$$\bar{\Gamma}^3_{13}=-\frac{1}{s-z_G}\frac{\partial z_G}{\partial \bar{x}^1}$$

with

$$\bar{\Gamma}^3_{21}=\bar{\Gamma}^3_{12}, \quad \bar{\Gamma}^3_{23}=\bar{\Gamma}^3_{32}, \quad \bar{\Gamma}^3_{13}=\bar{\Gamma}^3_{31}. \tag{13}$$

The individual contravariant components can be expressed in terms of the rectangular components as

$$\bar{\bar{u}}^1=\bar{u},$$

$$\bar{\bar{u}}^2=\bar{v},$$

$$\bar{\bar{u}}^3=\bar{u}\frac{\partial z_G}{\partial x}\left(\frac{\sigma-s}{s-z_G}\right)+\bar{v}\frac{\partial z_G}{\partial y}\left(\frac{\sigma-s}{s-z_G}\right)+\bar{w}\frac{s}{s-z_G}.$$

Using (12) and (13) in (8)–(10), it follows that this system of equations involves more nonzero terms than in the original specification in the rectangular system. The extra terms are particularly evident in the vertical equation in which the Christoffel symbols appear, and in the Coriolis terms. Clark (1977), and Gal-Chen and Somerville (1975) use (1) in equations similar to (8)–(10) in a nonhydrostatic meteorological model, although they do not retain the Coriolis terms.

At this point it is appropriate to introduce the hydrostatic assumption. From (4), it is evident that

$$\boldsymbol{\eta}^1\approx\mathbf{i}, \quad \boldsymbol{\eta}^2\approx\mathbf{j} \quad \text{and} \quad \boldsymbol{\eta}^3\approx\mathbf{k}\frac{\partial\sigma}{\partial z},$$

when

$$\left|\frac{\partial\sigma}{\partial x}\right| \ll \left|\frac{\partial\sigma}{\partial z}\right|, \quad \left|\frac{\partial\sigma}{\partial y}\right| \ll \left|\frac{\partial\sigma}{\partial z}\right|, \tag{14}$$

which permits (10) to be rewritten as

$$\frac{\partial\bar{\bar{u}}^3}{\partial t}=-\bar{\bar{u}}^j\frac{\partial\bar{\bar{u}}^3}{\partial\bar{x}^j}-\overline{\bar{u}^{j''}\frac{\partial\bar{u}^{3''}}{\partial\bar{x}^j}}-\bar{\Gamma}^3_{jl}\bar{\bar{u}}^j\bar{\bar{u}}^l-\bar{\Gamma}^3_{jl}\overline{\bar{u}^{j''}\bar{u}^{l''}}$$

$$-\bar{\theta}\left(\frac{\partial\sigma}{\partial z}\right)^2\frac{\partial\bar{\pi}}{\partial\bar{x}^3}-\frac{\partial\sigma}{\partial z}g-\hat{f}\frac{\partial\sigma}{\partial z}\bar{\bar{u}}^1, \tag{15}$$

as long as the magnitude of $\partial\bar{\pi}/\partial\bar{x}^3$ is at least as large as that of $\partial\bar{\pi}/\partial\bar{x}^1$ and $\partial\bar{\pi}/\partial\bar{x}^2$.

If the hydrostatic assumption is applied, where acceleration in the σ direction [which is essentially vertical as given by (14)] and the Coriolis terms are much less than the pressure gradient and the gravitational acceleration terms, then (15) reduces to

$$\frac{\partial\bar{\pi}}{\partial\bar{x}^3}=\frac{\partial\bar{\pi}}{\partial\sigma}=-\frac{g}{\bar{\theta}}\left(\frac{\partial\sigma}{\partial z}\right)^{-1}=-\frac{g}{\theta}\frac{s-z_G}{s}. \tag{16}$$

Similarly, (8) and (9) reduce to

$$\frac{\partial\bar{\bar{u}}^1}{\partial t}=-\bar{\bar{u}}^j\frac{\partial\bar{\bar{u}}^1}{\partial\bar{x}^j}-\overline{\bar{u}^{j''}\frac{\partial\bar{u}^{1''}}{\partial\bar{x}^j}}$$

$$-\bar{\theta}\frac{\partial\bar{\pi}}{\partial\bar{x}^1}+g\frac{\sigma-s}{s}\frac{\partial z_G}{\partial x}+\hat{f}\bar{\bar{u}}^3-f\bar{\bar{u}}^2, \tag{17}$$

$$\frac{\partial\bar{\bar{u}}^2}{\partial t}=-\bar{\bar{u}}^j\frac{\partial\bar{\bar{u}}^2}{\partial\bar{x}^j}-\overline{\bar{u}^{j''}\frac{\partial\bar{u}^{2''}}{\partial\bar{x}^j}}$$

$$-\bar{\theta}\frac{\partial\bar{\pi}}{\partial\bar{x}^2}+g\frac{\sigma-s}{s}\frac{\partial z_G}{\partial y}+f\bar{\bar{u}}^1. \tag{18}$$

Eqs. (16)–(18) are in the form obtained when the chain rule is applied separately for the horizontal and the vertical equations of motion. As shown here, however, (16)–(18) are only approximate relationships when a complete tensor transformation is applied, and they are valid only when (14) applies. The terms given in (14) can also be written as

$$\left|\frac{\partial\sigma}{\partial x}\right|_{\max}=\left|\frac{\partial z_G}{\partial x}\right|, \quad \left|\frac{\partial\sigma}{\partial y}\right|_{\max}=\left|\frac{\partial z_G}{\partial y}\right|.$$

Hence, the inequality given by (14) states that $|\partial z_G/\partial x| \approx |\partial z_G/\partial y| \ll 1$ is a necessary condition to assure the validity of (16)–(18). In terms of the terrain representation, this condition requires that *the slope must have an angle* $\ll 45°$.

The subgrid-scale terms which are included in (17) and (18) must also be parameterized in terms of known quantities in order to completely specify these equations. In the original rectangular coordinate system, it is the customary practice to decompose the subgrid-scale terms into vertical and horizontal components, such that, for the equation of motion with $i = 1$, for example,

$$-\bar{\rho}\overline{w''u''}=F_{z_u}, \quad -\bar{\rho}\overline{u^{j''}u''}=F^j_{H_u} \quad (j=1,2),$$

where F_{z_u} represents the vertical turbulent fluxes of the east–west, u, component of velocity, while $F^j_{H_u}$ indicates the horizontal turbulent fluxes of u (i.e., $-\bar{\rho}\overline{u''^2}$ and $-\bar{\rho}\overline{v''u''}$). This separation into two components in mesoscale models is necessitated for two major reasons:

• In most mesoscale models, the horizontal grid spacing (Δx, Δy) is much larger than the vertical spacing (Δz) so that the parameterizations of subgrid scale mixing in the horizontal and vertical directions would be expected to be quite different.

• Much more is known about the functional form of vertical subgrid scale fluxes than of horizontal subgrid-scale fluxes. Thus, two completely different parameterizations are required, with the vertical flux representation being much more detailed.

In a terrain-following coordinate system, when

$$\left|\frac{\partial z_G}{\partial x}\right| \approx \left|\frac{\partial z_G}{\partial y}\right| \ll 1,$$

it, therefore, is desirable to retain this separation into vertical and horizontal flux components. To illustrate this, multiply the first two terms on the right-hand side of the equality in (17) by $\bar{\rho}(s - z_G)/s$ so that

$$\bar{\rho}\,\frac{(s - z_G)}{s}\left[\bar{\tilde{u}}^j\,\frac{\partial \bar{\tilde{u}}^1}{\partial \tilde{x}^j} + \overline{\tilde{u}^{j''}\,\frac{\partial u^{1''}}{\partial \tilde{x}^j}}\right]$$

$$= \bar{\rho}\,\frac{(s - z_G)}{s}\left[\overline{(\bar{\tilde{u}}^j + \tilde{u}^{j''})\,\frac{\partial}{\partial \tilde{x}^j}\,(\bar{\tilde{u}}^1 + \tilde{u}^{1''})}\right],$$

$$= \frac{\partial}{\partial \tilde{x}^j}\,\bar{\rho}\,\frac{(s - z_G)}{s}\,\overline{(\bar{\tilde{u}}^j + \tilde{u}^{j''})(\bar{\tilde{u}}^1 + \tilde{u}^{1''})},$$

$$= \frac{\partial}{\partial \tilde{x}^j}\left[\bar{\rho}\,\frac{(s - z_G)}{s}\,\bar{\tilde{u}}^j\bar{\tilde{u}}^1\right] + \frac{\partial}{\partial \tilde{x}^j}\left[\bar{\rho}\,\frac{(s - z_G)}{s}\,\overline{\tilde{u}^{j''}\tilde{u}^{1''}}\right].$$

In writing this expression, the anelastic form of the conservation of mass equation, [i.e., $\partial(\rho u_j)/\partial x_j = 0$] in the transformed system,[3] given by

$$\frac{1}{s - z_G}\,\frac{\partial}{\partial \tilde{x}^j}\,\bar{\rho}(s - z_G)\bar{\tilde{u}}^j = 0,$$

along with the assumption that $\overline{u^{j''}} = \overline{u^{1''}} = 0$ as defined by (7), has been used. Similar terms, of course, can be derived for the advection terms in (18).

Thus, in the transformed coordinate system the subgrid-scale fluxes are given as

$$\bar{\rho}\,\frac{(s - z_G)}{s}\,\overline{\tilde{u}^{3''}\tilde{u}^{1''}} = F_{\sigma_{\tilde{u}^1}},$$

$$\bar{\rho}\,\frac{(s - z_G)}{s}\,\overline{\tilde{u}^{j''}\tilde{u}^{1''}} = F^j_{H_{\tilde{u}^1}} \quad (j = 1, 2)$$

where $F_{\sigma_{\tilde{u}^1}}$ and $F^j_{H_{\tilde{u}^1}}$ are, respectively, the $\tilde{u}^1$ fluxes in the $\tilde{x}^3$ direction, and the $\tilde{x}^1$ and $\tilde{x}^2$ directions.

Since it is assumed that

$$\left|\frac{\partial \sigma}{\partial x^1}\right| \approx \left|\frac{\partial \sigma}{\partial x^2}\right| \ll 1,$$

and so

$$\frac{\partial \sigma}{\partial z} \approx 1,$$

it is reasonable to also assume that the fluxes in the $\tilde{x}^3$ and x^3 directions in the two systems are almost equal and so

$$F_{z_u} \approx F_{\sigma_{\tilde{u}^1}} \approx \bar{\rho}\overline{w''u''}^R = \bar{\rho}\,\frac{s - z_G}{s}\,\overline{\tilde{u}^{3''}\tilde{u}^{1''}},$$

$$\overline{\tilde{u}^{3''}\tilde{u}^{1''}} = \frac{s}{s - z_G}\,\overline{w''u''}^R,$$

where the overbar with the R superscript is used to emphasize that this averaging volume is different from that given by (5) (in this case a rectangular volume). Moreover, if $\overline{w''u''}^R$ is assumed proportional to an exchange coefficient which is a function of height ξ above the ground and the mean velocity profile $\bar{u}^R$, as is often done, then

$$\overline{\tilde{u}^{3''}\tilde{u}^{1''}} \cong \frac{s}{s - z_G}\,\overline{w''u''} = -\,\frac{s}{s - z_G}\,K(\xi)\,\frac{\partial \bar{u}^R}{\partial z}.$$

Since $\bar{\tilde{u}}^1 = \bar{u}$, $\xi = \sigma(s - z_G)/s$ and $(\partial/\partial z) \approx s/(s - z_G) \times \partial/\partial\tilde{x}^3$, then this approximation for the vertical subgrid-scale flux becomes

$$\overline{\tilde{u}^{3''}\tilde{u}^{1''}} = -\left(\frac{s}{s - z_G}\right)^2 K\left(\sigma\,\frac{s - z_G}{s}\right)\frac{\partial \bar{\tilde{u}}^1}{\partial \tilde{x}^3},$$

and so the $\tilde{x}^3$ flux term in (17) can be represented as

$$\overline{\tilde{u}^{3''}\,\frac{\partial \tilde{u}^{1''}}{\partial \tilde{x}^3}} = \left(\frac{s}{s - z_G}\right)^2 \frac{\partial}{\partial \tilde{x}^3}\,K\,\frac{\partial \bar{\tilde{u}}^1}{\partial \tilde{x}^3},$$

where K is a function of $\sigma(s - z_G)/s$ (i.e., is a function of height above the ground). The subgrid flux in the $\tilde{x}^3$ direction in (18) can be shown to have the same form.

The subgrid-scale fluxes in the $\tilde{x}^1$ and $\tilde{x}^2$ directions could be written in a similar form; however, since essentially nothing is known about their functional form on the mesoscale in the rectangular coordinate representation, no purpose is served by writing them here. Subgrid-scale fluxes in the horizontal direction are included in models for computational reasons only.

3. Summary and discussion

This paper uses tensor transformation procedures in order to derive a terrain-following coordinate system which is frequently used in a number of regional and mesoscale hydrostatic models. The technique utilizes tensor transformation procedures in order to ensure the physical invariance of the con-

[3] See Dutton (1976, p. 143) for the derivation of this mass conservation relation in any holonomic coordinate system.

servation relations between the Cartesian and terrain-following systems.

The analysis has shown that, in general, applying the chain rule separately to the hydrostatic equation and the horizontal equations of motion in order to transform them to a generalized vertical coordinate system yields a different form of equation than when the tensor transformation is applied before the hydrostatic assumption is made. Only when the slope of the terrain is much less than 45° will the two procedures of obtaining transformed equations yield the same forms.

In addition, if the hydrostatic assumption is made before the coordinate transformation is applied, kinetic energy is generally calculated by the contravariant velocity components (i.e., $e = \frac{1}{2}\tilde{u}^i\tilde{u}^i$). As given by Dutton (1976, p. 250), however, this formulation is incorrect since kinetic energy should be derived from the product of the contravariant and covariant velocity components [(i.e., $e = \frac{1}{2}(\tilde{u}^i\tilde{u}_i)$]. Even for small slope angles, the use of the contravariant components alone could result in significant errors in kinetic energy calculations.

Moreover, because the equations must be averaged, in order to apply them to mesoscale and regional-scale atmospheric problems, variations of the gradient in the generalized vertical coordinate system within a grid volume must be small. When the generalized coordinate includes terrain elevation, this condition requires that the variations in the gradient of the actual topography within areas equivalent to the model grid mesh, must be much smaller than variations in the averaged terrain slope between adjacent grid mesh areas. Otherwise, terms such as the subgrid scale velocity fluxes must include the effects of correlations between terrain slope and the fluctuating velocities.

The Fourier decomposition of elevation over a region offers one methodology to examine terrain variability over a region. Pielke and Kennedy[4] (1980), have recently used such a spectral technique in order to determine the spatial scales of the terrain slopes in a portion of central Virginia. Such an evaluation is essential for the definition of the proper grid resolution required in a model (i.e., to assure that most variations of terrain slope are on a spatial scale larger than the grid increment in the model).

It was also shown that the effect of the generalized coordinate on the parameterization of the planetary boundary layer must be considered. In defining a profile exchange coefficient as a function of distance above the ground, for instance, the eddy coefficient must be computed from the actual height and not the new generalized vertical coordinate. Although boundary-layer theory over irregular terrain is not well advanced, and though errors in its proper representation using horizontally homogeneous, steady-state boundary-layer theory may be significant, one should at least be assured that its mathematical representation is consistent.

Finally, this study was undertaken in order to critically examine the use of a terrain-following coordinate system in hydrostatic models. Although the resultant equations usually can be written in the same form as those used in the past, this investigation critically delineates the conditions under which they apply. These conditions should be considered whenever simulations of atmospheric circulations are attempted.

[4] Pielke, R. A., and E. Kennedy: Mesoscale Terrain Features, January 1980, Rep. No. UVA-ENV SCI-MESO-1980-1, Department of Environmental Sciences, University of Virginia, Charlottesville.

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