Thermal compression waves. I: Total-energy transfer

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SUMMARY

Total energy is considered a fundamental quantity that is transferred poleward by the general circulation of the atmosphere. In this study we investigate the mechanism for the upscale transfer of total energy from a localized heat source, such as a thunderstorm. Using a fully compressible numerical model it is shown that thermal compression waves can effectively transfer total energy at the speed of sound. It is suggested that physical interpretations of the energy cycle should take into consideration this mechanism for transferring total energy.

1. INTRODUCTION

A poleward transport of total energy is required to balance the net radiational surplus at low latitudes and deficit at high latitudes. The magnitude of this energy transport has been estimated by Oort (1971) and Vonder Haar and Oort (1973). The generally accepted view is that the Hadley circulation plays a significant role in the latitudinal transfer of total energy in the tropics, whereas at mid latitudes baroclinic eddies are dominant. In observational studies the total-energy flux across a latitude circle is typically decomposed into contributions by transient eddies, stationary eddies and mean meridional circulations (White 1951a, b; Starr and White 1954; Oort and Peixóto 1983; among others). Although the large-scale flux of total energy has been the subject of numerous observational investigations a physical understanding of how it is transferred upscale from localized heat sources, such as thunderstorms, has yet to be achieved. This mechanism is investigated in this study. Using a numerical model it would appear a straightforward matter to examine how the perturbed total-energy field evolves in space and time from where energy is deposited from a small-scale heat source. In a recent two-dimensional numerical investigation of Florida thunderstorms (Nicholls et al. 1991a) significant temperature perturbations are associated with rapidly propagating thermally forced gravity waves. It was originally speculated that these thermally forced gravity waves could result in a transport of total energy. Since the gravity waves propagate rapidly (∼25 m s⁻¹), much faster than the advective motions that occur in the simulation, it was thought of some interest to determine if they are responsible for a total-energy transfer. However, results to be presented in this article indicate that these thermally forced gravity waves do not result in a flux of total energy away from the heat source. It is inferred that thermal compression waves may be responsible for a transfer of total energy, and a fully compressible model is used to test this hypothesis.

At this stage it is pertinent to give an idea of what is meant by a ‘thermal compression wave’; a more precise definition will be given later. When an air parcel is heated the pressure increases and the parcel expands, while the adjacent region is compressed. The pressure rise responsible for compressing the adjacent air does not remain in situ, but propagates at the speed of sound. We refer to this type of sound wave as a thermal compression wave to distinguish it from the more commonly studied mechanically forced high-frequency sound wave. Although their role in mass adjustment has been discussed to some extent (Tripoli and Cotton 1982; Anderson et al. 1985; Droegemeier and Wilhelm-

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son 1987; Nicholls 1987), their possible role in the transfer of total energy does not appear to have received much attention. This possibility becomes apparent when the internal energy per unit volume, $\rho c_v T$, is written as $c_v p/R$ by substituting from the ideal gas relation ($p = \rho RT$) where $\rho$ is the density, $c_v$ the specific heat at constant volume, $T$ the temperature, $p$ the pressure and $R$ the gas constant. Therefore, a compression wave with a positive pressure perturbation is a region with an enhanced internal energy. It is important to note that the conservative flux form of the total-energy equation used in large-scale budget studies involves the energy per unit volume. This is appropriate since the flux across a latitude circle from one volume into another is considered.

It is emphasized at the outset that the results of this study by no means invalidate the sound-proof equations used in many numerical models. Furthermore, the total-energy transfer associated with thermal compression waves does not involve significant atmospheric motions or temperature changes. The concept of available potential energy (Margules 1903; Lorenz 1955) is best suited to understanding the energy available for meteorologically significant atmospheric motions. Nevertheless, the role that thermal compression waves have in the transfer of conserved quantities such as mass and total energy should at least be recognized.

As stated above, this investigation was undertaken in order to determine the physical mechanism responsible for the upscale transfer of total energy. However, as it became evident that compression waves play an important role in this process it was decided to examine their properties in more detail. A limitation of the simulations discussed in this first paper is that the upper-boundary condition is a rigid lid. In Part II (Nicholls and Pielke 1994) this restriction is removed and simulations are run with a radiative upper-boundary condition. The role that thermal compression waves have in mass adjustment is investigated. Results demonstrate that the thermal compression waves responsible for horizontal transfer of mass are basically Lamb waves. Hydrostatic numerical models which retain Lamb waves should conserve mass and total energy fairly well. Additionally in Part II, the vertical transport of total energy in a thermally driven mixed layer is examined. The hypothesis is tested that the upward transport of total energy is accomplished by compression waves rather than buoyant thermals. The question is addressed as to what conserved quantity the eddy flux of sensible heat refers and why it is a good measure of the total-energy transfer? In light of these results the notion of 'heat transport' in a fluid is discussed.

In section 2 of this article the total-energy equation is presented. In section 3 an energy analysis is carried out for a gravity wave generated in a two-dimensional simulation of Florica convection which shows that it does not result in a net horizontal flux of total energy away from the heat source. It is also shown that the numerical model, which is not fully compressible (essentially anelastic for the experiments conducted in these papers), does not conserve total energy, and that if a heat source is prescribed within the model domain it actually results in the net energy decreasing. In section 4 an analysis of the energetics of gravity waves is carried out using the linearized anelastic equations. Bulk energy formulations of these equations are constructed in order to determine if a heat input will result in a corresponding increase in the total-energy field. This is not found to be the case. An approximate system of equations would not be expected to conserve total energy exactly. However, these results demonstrate that even approximate conservation does not occur. For completeness, wave energy is discussed and its rate of production contrasted with that for total energy. Analytic solutions are obtained for the one-dimensional thermally forced sound-wave equations in section 5. It is shown that total energy is nearly conserved for these equations, in contrast to the anelastic system, and that thermal compression waves transfer total energy at the speed of sound. Com-
parison is made between the transfer of total energy and wave energy by thermal
compression waves. In section 6 an experiment is carried out with a fully compressible
version of the Colorado State University–Regional Atmospheric Modelling System
(CSU–RAMS) to test the hypothesis that total-energy transfer from a localized heat
source, such as a thunderstorm, is accomplished by thermal compression waves.

2. The total-energy equation

The derivation of the total-energy equation can be found in many standard texts
(see for example Gill 1982). In this study the conventional definitions for the various
forms of energy are followed. The internal-energy equation is

\[
\frac{\partial}{\partial t}(c_v \rho T) + \nabla \cdot (c_v \rho T \mathbf{u} - \kappa \nabla T) = Q_v - \rho \nabla \cdot \mathbf{u} + \rho \epsilon
\]  

(1)

where, \( t \) is time, \( \mathbf{u} \) is the velocity vector, \( \kappa \) the thermal conductivity and \( Q_v \) is the heating
rate per unit volume. The term \( \rho \epsilon \) represents the conversion of mechanical energy into
internal energy, where

\[
\epsilon = \nu \left( \frac{\partial \mathbf{u}}{\partial x} \right)^2 + \left( \frac{\partial \mathbf{u}}{\partial y} \right)^2 + \left( \frac{\partial \mathbf{u}}{\partial z} \right)^2
\]

is the dissipation rate, \( \mu \) is the viscosity of the fluid and \( \nu = \mu / \rho \) is the kinematic viscosity.

The gravitational potential-energy equation is

\[
\frac{\partial}{\partial t}(\rho gz) + \nabla \cdot (\rho g z \mathbf{u}) = w g \rho
\]  

(2)

where \( w \) is the vertical component of velocity and \( g \) the gravitational constant.

The kinetic-energy equation is

\[
\frac{\partial}{\partial t}\left( \frac{1}{2} \rho u^2 \right) + \nabla \cdot \left( \left( \rho + \frac{1}{2} \rho u^2 \right) \mathbf{u} - \mu \nabla \left( \frac{1}{2} u^2 \right) \right) = -w g \rho - \rho \epsilon + \rho \nabla \cdot \mathbf{u}.
\]  

(3)

(Note that in order to simplify the notation, \( u^2 \) has been written in place of \( u \cdot u \nabla \mathbf{u} \) throughout.) Adding these equations the total-energy equation is obtained;

\[
\frac{\partial}{\partial t}\left( c_v \rho T + \rho gz + \frac{\rho u^2}{2} \right) + \nabla \cdot \left( \left( c_v \rho T + \rho gz + \frac{1}{2} \rho u^2 \right) \mathbf{u} + p \mathbf{u} - \kappa \nabla T - \mu \nabla \left( \frac{1}{2} u^2 \right) \right) = Q_v.
\]  

(4)

The heating rate \( Q_v \) can be decomposed into a radiative flux divergence and latent-heat
release by means of the pseudoadiabatic process, so that the total-energy equation becomes;

\[
\frac{\partial}{\partial t}\left( c_v \rho T + \rho gz + \rho L q + \frac{\rho u^2}{2} \right) + \nabla \cdot \left( \left( c_v \rho T + \rho gz + \rho L q + \frac{1}{2} \rho u^2 \right) \mathbf{u} + p \mathbf{u} + F - \kappa \nabla T - \mu \nabla \left( \frac{1}{2} u^2 \right) \right) = 0
\]  

(5)

where \( F \) is the radiative flux density, \( q \) the specific humidity and \( L \) the latent heat of
condensation. In large-scale energy-budget studies the hydrostatic approximation is made
and the flux across a vertical latitudinal wall is rewritten in the \((x, y, p)\) coordinate system
(Priestley 1949). Oort and Peixóto (1983) give the following expression for the flux across
a circle of latitude:
\[ \int_{z=0}^{\infty} \int_0^{p_s} \frac{\rho (c_p T + gz + Lq)}{v \cos \phi} \, dv \, dz = 2\pi R_e \cos \phi \int_0^{p_s} (c_p T + gz + Lq) u \frac{dp}{g} \] (6)

where the enthalpy has been introduced \((c_p T = c_v T + p/\rho)\) where \(c_p\) is the specific heat at constant pressure, \(R_e\) is the radius of the earth, \(\phi\) the latitude, \(v\) the meridional velocity and \(p_s\) the surface pressure. In this study the phase change of water will be regarded as leading to a heating term as in Eq. (4), rather than being included in the total-energy term as in Eq. (5). The reason for this is that the focus of this study is the mechanism for the transfer of sensible and potential energy rather than latent heat. This point will be discussed more fully in the next section.

In later sections idealized experiments are carried out for a prescribed heat source in a dry atmosphere. For this case, \(c_v \rho T\) in the local time derivative of Eq. (4) can be replaced by \(c_v \rho / R\), using the ideal gas law. It is convenient to consider perturbations of the internal and gravitational potential energies from the base-state energies \(c_v p_0 / R\) and \(\rho_0 g z\). Then substituting \(p = p_0(z) + p'\) and \(\rho = \rho_0(z) + \rho'\) into the internal and gravitational potential-energy terms within the local time derivative of Eq. (4) gives:

\[ \frac{\partial}{\partial t} \left( \frac{c_v p'}{R} + \rho' gz + \frac{\rho u^2}{2} \right) + \nabla \cdot \left( \left[ \frac{c_v}{R} p + \rho g z + \frac{1}{2} \rho u^2 \right] u + \rho u - \kappa \nabla T - \mu \nabla \left( \frac{1}{2} u^2 \right) \right) = Q_v. \] (7)

The introduction of perturbation energies from a base state in the local time derivative does not involve any approximation. For this system of equations linear solutions are gravity waves and compressional waves. The maximum rate at which information can travel is the speed of the fastest waves. The approach taken is to prescribe a heat source within the middle of the domain that has impervious upper and lower boundaries and examine the energy fields before the fastest-moving waves can reach the lateral boundaries. In this way fluxes of energy out of the domain do not have to be considered (note that the base state is taken to be motionless and does not result in any fluxes). The total energy within the domain in this case is:

\[ \int_0^v \left( \frac{c_v}{R} p' + \rho' gz + \rho \frac{u^2}{2} \right) dV = \int_0^v \int Q_v dV \, dt \] (8)

where \(V\) is volume and the subscript \(v\) in the integral sign indicates integration over the volume of the domain. Because of the flux conservative form of Eq. (7) it is straightforward to prescribe a heat source \(Q_v\), and examine the evolution of the perturbed total-energy field.

3. Energy analysis for simulations with the standard version of the RAMS

Figures 1(a), (b), (c) and (d) show, respectively, fields of perturbation horizontal velocity, vertical velocity, perturbation pressure and perturbation temperature for the numerically simulated gravity wave discussed in Nicholls et al. (1991a) and Nicholls et al. (1991b, hereafter referred to as NPC). The reader is referred to the first of these papers for details of the numerical model. During the morning, sea breeze circulations develop over the Florida peninsula. By the early afternoon, deep convection starts to develop at the sea-breeze fronts. The strongest convection occurs during the late afternoon as the sea-breeze fronts converge. The subsequent decay of this convection produces two deep, oppositely moving, gravity waves. The gravity wave shown in Fig. 1 is moving from left to right and has propagated \(\sim 200\) km from where it was produced by convection
Figure 1. Gravity wave produced in a numerical simulation of Florida convection. (a) Horizontal velocity perturbation (m s^{-1}) from the initial state. (b) Vertical velocity (cm s^{-1}). (c) Perturbation pressure (Pa). (d) Perturbation temperature (K).
two and a half hours earlier. It has the structure of a clockwise rotating roll. The central region is warm and a high-pressure perturbation exists aloft and a low-pressure perturbation near the surface. Downward motion at the leading edge produces adiabatic warming, whereas upward motion at the back edge leads to adiabatic cooling, which tends to return the temperature towards environmental values as the wave passes.

Figures 2(a), (b), (c) and (d) show, respectively, fields of perturbation internal energy, perturbation gravitational potential energy, kinetic energy and perturbation total energy for the gravity wave. Since density is not a predicted variable in the model, and is needed to calculate the gravitational potential energy, it is diagnosed from the predicted values of pressure and potential temperature, using the ideal gas law. The perturbation internal energy reflects the perturbation pressure shown in Fig. 1(c). The perturbation gravitational potential energy is mainly negative owing to warming (Fig.1(d)) which indicates a density decrease \( \rho'/\rho_0 \sim -T'/T_0 \). There is a maximum in the lower stratosphere due to a cool region. The kinetic energy is relatively small. The perturbation total energy reflects mainly the contributions of the internal and gravitational potential energies. The sum of the perturbation internal, gravitational potential and kinetic energies within the region shown in Fig. 1 is \( -2.4 \times 10^9 \) J m\(^{-1}\). Hence, the region occupied by the gravity wave is one of net negative perturbation energy. Therefore, this wave is not transporting total energy away from the simulated thunderstorm. In fact an energy budget for the whole domain indicates that there is no net increase in total energy even during the time intense convection is occurring.

As discussed in section 2, the latent-heating term can be rewritten in flux conservative form, resulting in an additional contribution to the total energy called the latent heat. In other words it can be regarded as a conversion of energy from one form to another rather than as a source of energy (see, for example, Van Mieghem (1973), section 12.2). This conversion takes place within the simulated thunderstorm where condensation and precipitation occurs. Therefore, the moisture in the domain decreases as latent heat is converted to sensible heat. A negative low-level moisture anomaly develops as the gravity wave forms and is propagated with it (not shown). Hence, the perturbation latent-heat energy is negative within the gravity-wave region. These considerations do not change the conclusion that total energy has not been conserved. The latent heating should result in a net increase in internal and potential energies, and yet this is not found to be the case.

In the remainder of this study, heat sources are prescribed in a dry atmosphere so that the complexities associated with phase changes do not have to be considered. A constant heating rate having the form,

\[
S = \frac{S_0 a^2}{(x^2 + a^2)} \sin \left( \frac{\pi z}{H} \right)
\]

is applied in a region which has a rigid lid at \( z = H \), the height of the domain. In order for the solution to be similar to that of the incompressible analytic model discussed by NPC, the rigid lid is chosen to be low, at \( H = 2 \) km. The value of the half-width \( a \) is 10 km and \( S_0 = 0.0005 \) degC s\(^{-1}\). The heating rate per unit volume is \( Q_a = \rho c_p T S/\theta \), where \( \theta \) is potential temperature, giving a magnitude of \( \sim 0.5 \) J m\(^{-3}\) s\(^{-1}\). The way in which the prescribed heating rate is applied to the thermodynamic equation used in the RAMS is described in appendix A. The horizontal and vertical grid increments are 4 km and 100 m, respectively, and the width of the domain is 400 km. Results for the fields of horizontal velocity, vertical velocity, perturbation pressure and perturbation temperature at 0.5 h are shown in Figs. 3(a), (b), (c) and (d) respectively. There is flow towards the heat source at low levels and away from it aloft. Upward motion occurs at the centre of
Figure 2. Energy fields for the gravity wave. (a) Perturbation internal energy (J m$^{-3}$). (b) Perturbation gravitational potential energy (J m$^{-3}$). (c) Kinetic energy (J m$^{-3}$). (d) Perturbation total energy (J m$^{-3}$).
Figure 3. Numerical solution of (a) for rigid lid and (b) for prescribed heat source using the standard RAMS model. (a) Horizontal velocity (contour interval in 0.1 m/s). (b) Vertical velocity (contour interval in 0.1 m/s). (c) Perturbation pressure (contour interval is 4 Pa). (d) Perturbation temperature (contour interval is 0.2 K). Inward pointing ticks in these figures indicate grid points at 0.5 s intervals.
the source and compensating subsidence outside. There is a pressure low at the surface, a high aloft and a broad region of warming. The subsidence regions are travelling away from the centre of the source with a speed \( c = NH/\pi \approx 7 \text{ m s}^{-1} \), where the buoyancy frequency \( N = 10^{-2} \text{ s}^{-1} \). Further discussion of the characteristics of thermally forced gravity waves can be found in NPC, Lin and Smith (1986) and Bretherton (1988).

Fields of perturbation internal energy, perturbation gravitational potential energy, kinetic energy and perturbation total energy are shown in Figs. 4(a), (b), (c) and (d) respectively. Similarly to the gravity wave produced in the Florida convection simulation, the total energy reflects mainly the contributions due to internal and gravitational potential energies. The total perturbation energy in the domain is \(-1.1 \times 10^{7} \text{ J m}^{-1}\). Hence, although an energy source has been prescribed within the model domain it results in a net decrease in total energy. This shows that the standard version of the RAMS does not conserve total energy since Eq. (7) is not satisfied.

4. ENERGY ANALYSIS OF THE ANELASTIC SYSTEM

In the preceding section it was shown that the numerical model does not conserve total energy. In this section it is shown that using bulk energy formulations this result is also generally true for the linearized hydrostatic anelastic equations, for constant buoyancy frequency \( N \) and using what appear to be reasonable boundary conditions. The rate of production of gravity-wave energy is also discussed.

The linearized anelastic equations for small perturbations from a rest state are:

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial x} = 0 \tag{10}
\]

\[
\frac{\partial w}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{gp'}{\gamma \rho_0} - b = 0 \tag{11}
\]

\[
\frac{\partial b}{\partial t} + wN^2 = \frac{gQ_m}{c_p T_0} \tag{12}
\]

\[
\frac{\partial}{\partial x} \left( \rho_0 u \right) + \frac{\partial}{\partial z} \left( \rho_0 w \right) = 0 \tag{13}
\]

where the buoyancy \( b = g \theta'/\theta_0 \), \( Q_m \) the heating rate per unit mass \( (Q_m/\rho_0) \), and \( \gamma = c_p/c_v \). The reference-state variables, \( \rho_0, p_0, T_0 \), and \( \theta_0 \) may be functions of \( z \). Ogura and Philips (1962) and Dutton and Fichtl (1969) show that this system of equations is valid for deep convection. Making the hydrostatic approximation and assuming a constant buoyancy frequency \( N \), Eqs. (10)–(13) reduce to a single equation for perturbation pressure,

\[
\frac{\partial^2 p'}{\partial t^2} + \frac{\partial}{\partial z} \left( g \rho_0 p' \right) + \frac{\partial^2}{\partial z^2} \left( \frac{g \rho_0 p'}{\gamma \rho_0} \right) + N^2 \frac{\partial^2 p'}{\partial x^2} = \frac{\partial}{\partial t} \left( \frac{\rho_0 g Q_m}{c_p T_0} \right). \tag{14}
\]

The objective is to obtain an expression for the rate of change of internal and gravitational potential energies and see how this is related to the heating rate. The perturbation internal energy within a column is

\[
E_i = \int_{0}^{\infty} \frac{c_p}{R} p' \, dz. \tag{15}
\]

Integrating Eq. (15) with respect to \( z \), from the surface to \( z = \infty \), gives
Figure 4. Energy fields for the standard RAMS model and a prescribed heat source. (a) Perturbation internal energy (contour interval is 8 J m⁻²). (b) Perturbation gravitational potential energy (contour interval is 8 J m⁻²). (c) Kinetic energy (contour interval is 0.08 J m⁻²). (d) Perturbation total energy (contour interval is 8 J m⁻²).
\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial p'}{\partial z} \right)_{z=\infty} - \left( \frac{\partial p'}{\partial z} \right)_{z=0} + \frac{\partial^2}{\partial t^2} \left( \frac{\rho g p'}{\gamma p_0} \right)_{z=\infty} - \left( \frac{\rho g p'}{\gamma p_0} \right)_{z=0} +
\]
\[
N^2 \int_0^\infty \frac{\partial^2 p'}{\partial x^2} \, dz = \frac{\partial}{\partial t} \left( \frac{\rho g Q_m}{c_p T_0} \right)_{z=0}
\]
(16)

where it has been assumed that the heating does not extend to \( z = \infty \). From Eq. (12) and the hydrostatic version of Eq. (11), it can be shown that
\[
\frac{\partial}{\partial t} \left( \frac{\rho g Q_m}{c_p T_0} \right)_{z=0} = \frac{\partial^2}{\partial t^2} \left( \frac{\rho g p'}{\gamma p_0} \right)_{z=0} + \frac{\partial^2}{\partial t^2} \left( \frac{\partial p'}{\partial z} \right)_{z=0}.
\]
(17)

If it is assumed that the perturbation pressure and its vertical gradient for large \( z \) become negligibly small, then Eq. (16) becomes
\[
\frac{\partial}{\partial x} \int_0^\infty p' \, dz = \text{constant.}
\]
(18)

Using the boundary condition that \( \partial p'/\partial x = 0 \) at \( x = \pm \infty \), which would seem reasonable if heating is turned on at some time within a localized region, this constant is zero. Integrating Eq. (18) with respect to \( x \) and using the condition that \( p' = 0 \) at \( x = \pm \infty \), gives
\[
\int_0^\infty p' \, dz = 0
\]
(19)
everywhere within the domain. Therefore, \( E'_i = 0 \) everywhere and for all time. For a hydrostatic atmosphere it is straightforward to show that the perturbation gravitational potential energy \( E'_p = RE'_i/c_o \), which must also be zero.

Hence, for the conditions stated earlier, there is no change in the internal and gravitational potential energies within a column. This result implies that total energy is not conserved in the following sense; if the perturbed pressure field was used to calculate the sum of the perturbed internal and gravitational potential energies for the primitive-equation system it would be found to be approximately equal to the heat input, since the kinetic-energy contribution is small. However, for the linearized hydrostatic anelastic system there is no corresponding equivalence between the sum of the perturbed internal and gravitational potential energies, as determined from the perturbed pressure field, and the heat input. Since this is an approximate system, exact agreement would not be expected, but apparently not even an approximate equivalence is obtained. For the rigid-lid case it can be shown that the perturbation internal energy within a column is again zero. However, the perturbation gravitational potential energy is,
\[
E'_p = \int_0^H \rho' g z \, dz = -p'(H)H
\]
(20)
where the perturbation hydrostatic equation \( \partial p'/\partial z = -\rho'g \) and integration by parts has been used. This result is consistent with the energy analysis for the numerical experiment with a prescribed heat source and a rigid lid, where it was found that the perturbation gravitational potential energy decreased. For a semi-infinite region the warming occurring in the heated region leads to a decrease in density and hence in gravitational potential energy; however, above the heat source cooling occurs, leading to an increase in gravitational potential energy (see Fig. 6(d) NPC). The result is that there is no net change of the gravitational potential energy in a column.
The incompressible Boussinesq equations, valid for shallow convection, differ from Eqs. (10)–(13) in that the term \( gp'/(\gamma p) \) is neglected in the vertical-momentum equation (Eq. 11), the buoyancy is replaced by \( gT'/T_0 \) and the incompressible continuity equation is used. For this system of equations the following wave-energy equation can be formed:

\[
\frac{\partial}{\partial t} \left( \rho_0 \left( \frac{u^2 + w^2}{2} \right) \right) + \frac{\partial}{\partial x}(\rho_0 b^2) + \frac{\partial}{\partial z}(\rho_0 b Q_m) = \frac{\rho_0 g b Q_m}{c_p T_0 N^2}. \tag{21}
\]

The first term within the local time derivative can be identified as the kinetic energy; the second term is called the wave potential energy. It is not possible to obtain an equation for wave energy for the deep-convection equations, Eqs. (10)–(13), without making an approximation (Dutton and Fichtl 1969). If acoustic modes are not filtered from the linearized system of equations a wave-energy equation can be obtained without the approximation (Eckart 1960). As pointed out by Dutton and Johnson (1967), the wave potential energy is basically a perturbation form of the available potential energy (Margules 1903; Lorenz 1955). The wave energy in Eq. (21) is positive definite and its rate of change depends on the correlation between \( b \) and \( Q_m \). This equation is not a conservative energy equation in the same sense as Eq. (4) or Eq. (7) for which an energy source or sink leads to corresponding changes in the energy field (see discussion by Dutton and Fichtl (1969), or in terms of the generation of available potential energy, Lorenz (1955) and Johnson (1970)). For instance, for the model discussed in NPC a positive heating rate leads to a positive buoyancy and an increase in total wave energy. Whereas cooling leads to a negative buoyancy and again an increase in total wave energy. In addition to the fact that the change in wave energy can be positive for both heat input and output, the rate of increase of the wave energy is considerably less than the magnitude of the heating rate. This is also a well-known phenomenon in large-scale studies of the generation of available potential energy. It can be demonstrated by integrating Eq. (21) over a volume and assuming that there are no fluxes across the boundaries, giving

\[
\frac{\partial}{\partial t} \int_v \left( \frac{\rho_0 (u^2 + w^2)}{2} + \frac{\rho_0 b^2}{2N^2} \right) \, dV = \int_v \frac{\rho_0 g b Q_m}{c_p T_0 N^2} \, dV. \tag{22}
\]

This is the rate of increase of wave energy in the domain. On the other hand, the rate at which energy is released in the domain is

\[
\int_v \rho Q_m \, dV. \tag{23}
\]

Comparing these two rates,

\[
\int_v \frac{g^2 T_r \rho_0 Q_m}{c_p T_0 N^2} \, dV; \quad \int_v \rho Q_m \, dV \quad \tag{24}
\]

the factor \( g^2/(c_p T_0^2 N^2) \sim 1/100 \) for \( N = 10^{-2} \). Since in convective systems \( T_r \sim 3 \text{ degC} \), the rate at which wave potential energy is increasing seems to be a factor of \( \sim 30 \) smaller than the rate at which heat is being released. This is a qualitative estimate, particularly since Eq. (21) is only a good approximation for shallow convection.

The considerations of this section indicate that for the anelastic system a prescribed heat input does not lead to a corresponding increase in the total-energy field. A qualitative estimate demonstrates that the production of wave energy for a small-scale heat source, such as a convective cloud system, proceeds at a much slower rate than the heat input. This estimate, which is in terms of the production rate of wave potential energy by a
small-scale heat source, is consistent with large-scale observations of available potential energy, which is found to be a small fraction of the total energy of the atmosphere. This is also consistent with the concept of convective available potential energy (CAPE), a measure of the energy available to a thunderstorm which can be realized as kinetic energy. CAPE is discussed further in section 7. The question remains as to why the anelastic system of equations does not conserve total energy. One would expect that a heat source would result in an increase in internal energy and gravitational potential energy in the vicinity of the source. However, the system of equations discussed so far does not seem to be able to describe this process. They do, however, seem to predict accurately the production of energy available for meteorologically significant atmospheric motions. Thermal compression waves are eliminated from this system of equations, and in the next section the possibility that they can accomplish a total-energy transfer is investigated.

5. THERMAL COMPRESSION WAVES

The one-dimensional linearized momentum, thermodynamic and continuity equations which allow the existence of thermal compression waves are, respectively,

\[ \frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial x} = 0 \]  \hspace{1cm} (25)

\[ \frac{\partial p'}{\partial t} - c^2 \frac{\partial^2 p'}{\partial x^2} = \frac{\gamma p_0}{c_p T_0} Q_m \]  \hspace{1cm} (26)

\[ \frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \frac{\partial u}{\partial x} = 0 \]  \hspace{1cm} (27)

where \( c = \sqrt{(\gamma RT_0)} \) is the speed of sound and \( \rho_0 \) is assumed constant. These reduce to a single equation for pressure:

\[ \frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = \frac{\gamma p_0}{c_p T_0} \frac{\partial Q_m}{\partial t} \]  \hspace{1cm} (28)

Consider a heating function \( Q_m = U(t - t_0)Q_o\alpha^2/(x^2 + a^2) \) where the Heaviside unit function \( U(t - t_0) = 1 \) for \( t > t_0 \) and 0 for \( t < t_0 \). Taking Laplace and Fourier transforms gives

\[ s^2 \tilde{p} + c^2 k^2 \tilde{p} = \exp(-t_0 s) \frac{\gamma p_0}{c_p T_0} Q_o \alpha a \exp(-a |k|) \] \hspace{1cm} (29)

where \( s \) and \( k \) are the parameters associated with the Laplace and Fourier transforms, respectively, and \( (\cdot) \) represents both the Laplace and Fourier transforms. The solution for \( \tilde{p} \) is

\[ \tilde{p} = \frac{\exp(-t_0 s) \gamma p_0 Q_o \alpha a \exp(-a |k|)}{c_p T_0 (s^2 + c^2 k^2)} \] \hspace{1cm} (30)

Taking the inverse Laplace and Fourier transforms the solution for \( t_0 = 0 \) (i.e. the heating immediately commences) is

\[ p'(x, t) = \frac{\gamma p_0 Q_o a}{c_p T_0 c} \frac{1}{2} \left[ \arctan \left( \frac{ct + x}{a} \right) + \arctan \left( \frac{ct - x}{a} \right) \right] \] \hspace{1cm} (31)
\[ u(x, t) = -\frac{\gamma p_0 Q_0 \alpha}{\rho_0 c_p T_0 c^2} \frac{1}{2} \left( \arctan \left( \frac{ct + x}{a} \right) - \arctan \left( \frac{ct - x}{a} \right) - 2 \arctan \left( \frac{x}{a} \right) \right) \]

(32)

\[ \rho'(x, t) = \frac{p'}{c^2} = \frac{\gamma p_0 Q_m t}{c_p T_0 c^2}. \]

(33)

The solution for a heating rate of 1 J kg\(^{-1}\) s\(^{-1}\), \(a = 20\) km and \(t = 600\) s is sketched in Fig. 5 (\(T_0 = 300\) K, \(p_0 = 10^5\) Pa). The pressure rapidly increases with the passage of the wave front and approaches a constant value given by \((\gamma p_0 Q_0 / c_p T_0 c)(\pi / 2)\). Equation (33) for the perturbation density shows that at the centre of the source the density soon starts to decrease linearly with time since the perturbation pressure approaches a constant value.

The air in this region is expanding. Far from the centre of the source the air is initially compressed with the passage of the wave front. Note that because of the particular form of the heating function chosen, which does not decrease to zero at large (but finite) \(\pm x\), the second term eventually dominates in Eq. (33).

The solution for the case where the heating is turned off after some time \(t^*\) is given in appendix B. The solution at \(t = 1200\) s for \(t^* = 600\) s is sketched in Fig. 6 (the values for the other parameters are the same as used in Fig. 5). Two oppositely moving compression waves are formed. The pressure within the heat-source region returns to the ambient value. However, the perturbation density within the heat-source region remains permanently lowered. Consequently, the air is also permanently warmer.

We will refer to the solutions of the nonhomogeneous wave equation (Eq. 28) as thermal compression waves. This emphasizes that they are generated by diabatic effects in contrast to mechanical forcing. The more commonly studied mechanically forced sound waves are usually high-frequency periodic waves. The compression waves studied

![Figure 5](image_url)

**Figure 5.** Analytic solution for a horizontally propagating thermally forced compression wave.
in this paper lead to net fluxes of mass and total energy away from their source. A wave is an identifiable signal or disturbance in a medium that is propagating in time, carrying energy with it. This is an accurate description of the solutions shown in Figs. 5 and 6, although by 'energy' is meant wave energy, rather than total energy. Equation (28) also admits solutions which arguably do not meet the strict definition of a wave. For instance, if one side of a box containing an ideal gas is heated this will cause the gas to expand in the heated region, while the gas on the other side of the box is compressed. This process is described by Eq.(28) with suitable boundary conditions, yet one may not wish to consider the compression of the unheated region as being associated with a wave. On the other hand, with the commencement of heating a pressure wave will be generated, which will subsequently rapidly reflect back and forth off the sides of the box. Each time the wave front passes a fixed point a small jump in pressure will occur. Since these pressure jumps are so small and so frequent it looks like a continuous process. Furthermore, information is propagated at the speed of sound; any fluctuation of the heat source will create a disturbance travelling at this speed. Whether or not all the solutions of Eq. (28) meet the strict criteria of being a wave, the important point is that compression results in a flux of mass and total energy from one volume into another.

Proceeding in the same manner as for the linearized anelastic system, the perturbation internal energy associated with the compression wave is found by integrating Eq. (28) with respect to \(x\):

\[
\frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} p' \, dx - c^2 \left( \frac{\partial p'}{\partial x} \right)_{-\infty}^{\infty} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{\gamma p_0}{c_p} T_0 Q' \, dx.
\]

Using the boundary condition \(\frac{\partial p'}{\partial x} = 0\) at \(x = \pm \infty\) the second term on the left side of

![Figure 6. Analytic solution for a thermal compression wave produced by a pulse forcing.](image-url)
Eq. (34) is zero. For a heating which commences at some time $t > 0$ (i.e. $Q_m(t = 0) = 0$), successive integration with respect to time and use of Eqs. (26) and (27) at $t = 0$ gives:

$$\int_{-\infty}^{\infty} \frac{c_p}{R} p' \, dx = \int_{0}^{\infty} \int_{-\infty}^{\infty} \rho_0 Q_m \, dx \, dt.$$  \hfill (35)

Hence the perturbation internal energy is equal to the energy input. For energy to be precisely conserved the kinetic energy also has to be added to the perturbation internal energy. However, the kinetic energy is only a small fraction of the perturbation internal energy and this small error is a result of linearizing the equations. This result, that energy is nearly conserved for the linearized equations which have compression-wave solutions, is in contrast to what was found for the linearized anelastic system.

In the atmosphere the energy transferred by high-frequency periodic sound waves, which result from mechanical vibrations, is small. This refers explicitly to wave energy, but it is also true that these waves do not result in a significant transfer of total energy. The wave-energy equation for compression waves, derived from Eqs. (25)–(27), is

$$\frac{\partial}{\partial t} \left( \frac{\rho_0 u^2}{2} + \frac{p'^2}{2\rho_0 c^2} \right) + \frac{\partial}{\partial x}(u p') = \frac{p' Q_m}{c_p T_0}.$$  \hfill (36)

Similarly to the gravity-wave case it can be shown that the rate of increase of wave energy is much smaller than the rate of heat release for typical atmospheric conditions. However, unlike periodic sound waves, thermal compression waves can result in a significant transfer of total energy. When a mechanically forced sound wave passes a fixed point, a material element of the fluid will undergo oscillations about that point, finally coming to rest in the same position. No net work is done against the base-state pressure, which appears to be the reason for not considering it in the energetics of sound waves (Lighthill 1978). However, if a thermal compression wave produced by a net heat input (such as shown in Fig. 6) passes a fixed point, a material element will come to rest in a different position. A considerable amount of work is done against the base-state pressure as the waves pass, which results in a net transfer of total energy.

As an example of the difference in magnitude between wave energy and perturbation internal energy, consider compressing air in a container at an initial pressure $p_0 = 1000 \text{ hPa}$ until its pressure increases by $0.1 \text{ hPa}$ and its volume is $V$. Suppose this compressed air is interchanged with an equal volume of air in a channel at pressure $p_0$, as sketched in Fig. 7(a). The wave potential energy is

$$\frac{p'^2 V}{2 \rho_0 c^2} \sim 4 \times 10^{-4} V$$  \hfill (37)

for $\rho_0 = 1 \text{ kg m}^{-3}$, whereas the perturbation internal energy is

$$\frac{c_p}{R} p' V \sim 25V.$$  \hfill (38)

The perturbation internal energy for this case is over four orders of magnitude larger than the wave potential energy. The instantaneous removal of the partitions leads to an initial-value problem for the perturbation pressure governed by Eq. (28), except that there is no heating term for this case. The solution to this type of problem is well known and sketched in Figs. 7(b) and (c). Two oppositely moving waves are formed, each having a perturbation pressure of $p'/2$ and moving at the speed of sound. The wave potential energy in each disturbance is a factor of four less than that in the initial disturbance. Therefore, the total wave potential energy has decreased by a factor of two and this is
concurrent with an increase in kinetic energy of the same magnitude. On the other hand the perturbation internal energy is a linear function of the perturbation pressure, so that each disturbance has half as much perturbation internal energy as the initial disturbance. Therefore, there is no change in the net internal energy. Obviously the internal energy must have decreased slightly because there has been an increase in kinetic energy. This small error must be a result of linearizing the equations.

Moreover, if the air in the container was initially cooled by adiabatic expansion it would lead to a similar solution, except that the perturbation pressure and velocities would be reversed in sign. Hence, the perturbation internal energy would be negative, although the wave potential energy would still be positive. In these examples the density of the gas that was compressed or expanded returns to that of the ambient state. However, if instead the gas was heated/cooled at a fixed volume and then introduced into a channel and the partitions removed, the resultant expansion/compression would lead to density changes, as occurs for the solutions shown in Figs. 5 and 6.

Temperature perturbations also occur in the compression waves shown in Fig. 6. Far away from the center of the heat source $p' \sim c^2 \rho'$. Substituting into the perturbation form of the equation of state,

\[
\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0}
\]

(39)
gives

\[
T' = \frac{RT_0}{c_0 \rho_0} \rho'
\]

(40)
which is the magnitude of the temperature perturbation within the wave of compression. In the heated region, which has been left permanently warmer $(T'' \sim - T_0 \rho'/\rho_0)$, the energy per unit mass $(c_p T)$ has increased, whereas the energy per unit volume $(\rho c_p T)$ returns to the ambient value.

It is evident that thermal compression waves can transfer total energy $(\rho c_p T + \rho u^2 / 2)$, for this case) at the speed of sound, from one volume into another. It is important to realize that the transfer of total energy by compression waves does not involve significant temperature changes. Heat transfer within a fluid is often considered to be associated with temperature changes. However, the conserved quantity total energy involves energy per unit volume (which depends on density as well as temperature), not energy per unit mass. In the next section a numerical model is used to investigate the transfer of total energy in a system which allows the existence of both gravity waves and compression waves. The hypothesis is tested that thermal compression waves are responsible for the upscale transfer of total energy from a localized heat source.

6. A NUMERICAL SIMULATION USING A FULLY COMPRESSIONABLE MODEL

Although the arguments that have been presented concerning the energetics of gravity waves and compression waves are suggestive it is by no means obvious what will happen in a system which allows the existence of both wave types. Since an analytical approach appears to be intractable this problem is addressed by using a numerical model.

The standard version of the RAMS employs a pressure-tendency equation, given in tensor notation by:

$$\frac{\partial \pi'}{\partial t} + \frac{c^2}{c_p \rho \theta'_v} \frac{\partial}{\partial x'_j} (\rho \theta'_v u'_j) = f'_\pi$$

(41)

where

$$f'_\pi = -u'_j \frac{\partial \pi'}{\partial x'_j} + \frac{\pi'}{c_v} \frac{\partial u'_j}{\partial x'_j} + \frac{c^2}{c_p \theta'_v} \frac{\partial \theta'_v}{\partial t} + D'_\pi$$

(42)

and $\pi'$ is the perturbation non-dimensional pressure from the initial state $\pi = (p/p_0)^{R/c_p}$. $\theta'_v$ is the virtual potential temperature and $D'_\pi$ is the diffusion. As discussed by Klemp and Wilhelmson (1978) the terms in $f'_\pi$ do not appear to have a significant influence on the simulation of a storm, and for computational efficiency are typically neglected. However, the term involving $\partial \theta'_v/\partial t$ would appear to be crucial for allowing expansion of a gas due to heat release, and hence the formation of thermally forced compression waves. Klemp and Wilhelmson found that the only noticeable difference when the terms in $f'_\pi$ were included was small shifts in the mean pressure in the domain, but that the pressure gradients remained the same.

The model has been modified so that the density is predicted from the full compressible continuity equation and the non-dimensional pressure is then diagnosed. A centre-in-space leap-frog scheme is used to solve the compressible continuity equation. Unlike the standard version of the RAMS no time-splitting scheme is used to solve the equations for the faster-moving sound-wave modes; all equations are solved using a short time step. The gravitational constant and the total vertical-pressure gradient are used in the vertical-momentum equation so that no linearized approximation to the buoyancy is made.

The design of the experiment is as follows: the domain is two-dimensional with length 2000 km and height 10 km. The horizontal grid increment is 5 km and the vertical
grid increment is 1 km. A heat source having the spatial distribution given by Eq. (9) is placed in the centre of the domain. The value of \( a \) is 10 km and the magnitude of the heating rate is 0.002 degC s\(^{-1}\). This heating rate is applied for 20 minutes and then turned off. The total simulation time is 40 minutes.

During the first few seconds of the simulation, expansion of the air occurs away from the heat source both upwards and downwards as well as in the horizontal. However, the effect of the rigid lid and the surface is to act as a wave guide which quickly channels the compression-wave motion into the horizontal. The results for fields of horizontal velocity, vertical velocity, perturbation temperature and perturbation pressure at 20 minutes are shown in Figs. 8(a), (b), (c) and (d) respectively. The gravity wave is evident in the \( u \) field between \( x = \pm 100 \) km. Notice that broad regions of weak flow (\( \sim 4 \) cm s\(^{-1}\)) have reached \( x = \pm 400 \) km. This is the compression wave. Significant vertical velocities are confined to the gravity-wave region. The perturbation pressure field shows a high aloft and a low near the surface between \( x = \pm 75 \) km, as would be expected from the gravity-wave solution. However, superimposed on this is a broad homogeneous region of increased pressure \( \sim 0.15 \) hPa in magnitude. Significant temperature increases are confined to the gravity-wave region.

Figures 9(a), (b), (c) and (d) show fields of perturbation internal energy, perturbation gravitational potential energy, kinetic energy and perturbation total energy, respectively, at \( t = 20 \) minutes. Within the gravity wave there is a large increase in internal energy at upper levels and a decrease at lower levels. A broad region of weak energy density extends to \( x = \pm 400 \) km within the compression wave. The gravitational potential energy is negative within the warm region of the gravity wave. There is a weak increase in gravitational potential energy within the compression wave and this is largest at upper levels. Kinetic energy is largest where the horizontal component of velocity is largest. It is considerably smaller than the other contributions to the energy budget. The net perturbation total energy within the gravity wave can be seen to be negative. The lower-level decrease in energy outweighs the upper-level increase. Outside the gravity wave there is a broad region of energy increase.

Figures 10(a), (b), (c) and (d) show fields of horizontal velocity, vertical velocity, perturbation pressure and perturbation temperature, respectively, at \( t = 40 \) minutes. From the horizontal-velocity field it can be seen that the gravity wave has separated into two oppositely moving waves, similar to the solution discussed by NPC. This has also occurred for the compression-wave mode which has formed two waves situated between \( x = 400 \) km and \( x = 800 \) km and \( x = -400 \) km. The separation of the gravity-wave mode is further evident in the other fields. The perturbation pressure is high within the compression wave.

Figures 11(a), (b), (c) and (d) show fields of perturbation internal energy, perturbation gravitational potential energy, kinetic energy and perturbation total energy, respectively, at \( t = 40 \) minutes. The perturbation internal energy within the gravity waves is positive aloft above symmetrical negative regions. Within the compression wave, perturbation internal energy is positive. The perturbation gravitational potential energy is negative in the gravity waves and positive in the compression waves. The kinetic energy is small and concentrated within the gravity wave. The net perturbation total energy within the gravity wave is negative and in the compression wave it is positive.

The total energy input into the domain during the 20-minute heating period was \( 3.143 \times 10^{11} \) J m\(^{-1}\). The total energy within the domain determined from the sum of the perturbation internal, perturbation gravitational potential and kinetic energies is \( 3.106 \times 10^{11} \) J m\(^{-1}\) at 20 minutes and \( 3.104 \times 10^{11} \) J m\(^{-1}\) at 40 minutes. Hence, the energy input and the energy measured in the domain are in reasonable agreement.
Figure 8. Solutions for the primitive-equation model at 20 minutes. (a) Horizontal velocity (contour interval is 8 cm s⁻¹). (b) Vertical velocity (contour interval is 0.2 K). (c) Perturbation pressure (contour interval is 10 Pa). (d) Perturbation temperature (contour interval is 0.2 K).
Figure 9. Energy fields for the primitive-equation model at 20 minutes. (a) Perturbation internal energy (contour interval is 12 J m⁻²). (b) Perturbation gravitational potential energy (contour interval is 20 J m⁻²). (c) Kinetic energy (contour interval is 0.12 J m⁻²).
Figure 10. As in Fig. 8 but for the primitive-equation model at 40 minutes.
Figure 11. As in Fig. 9 but for the energy fields for the primitive equation model at 40 minutes.
The parametrized diffusion in the model which is used to control numerical noise will result in some energy loss from the domain through the dissipative term of Eq. (3). However, since the deformation are very small in this simulation, the eddy viscosities are correspondingly so, resulting in a very slow dissipation of energy. In the real atmosphere the dissipation of kinetic energy will result in an increase in internal energy which is represented in Eq. (1). The model does not include this conversion of kinetic energy to internal energy so it is not energy conserving. It would be possible to add this term to the thermodynamic equation but it would not be physically realistic since the diffusion rates which occur in the model are typically much larger than actually occur in the atmosphere.

The analytic solution for the one-dimensional thermally forced compression-wave equation (Eqs. (31)–(33)) shows that density begins to decrease linearly with time within the heat-source region (Eq. (33)). However, for a system allowing gravity waves, the density does not keep decreasing in the heat-source region. It initially decreases, but then approaches a constant value within a region that is expanding at the gravity-wave speed. Compensating subsidence resulting in the replacement of air by less dense potentially warmer air from aloft leads to a density reduction in an increasingly large region.

For the rigid-lid case the net energy within the region occupied by the gravity wave becomes negative. This means that the energy transport away from the source by the compression wave is actually greater than the heat input. The rigid lid acts as a wave guide so that the motion in the compression wave is essentially horizontal at right-angles to the gravitational field. The expansion of air which takes place within the gravity wave leads to a decrease in gravitational potential energy, whereas outside this region compression leads to an increase in gravitational energy which is much smaller in magnitude but occurs over a much larger area.

When the experiment was repeated with the standard version of the RAMS there was no discernible difference at \( t = 40 \) minutes from that of the primitive-equation simulation within the gravity-wave region.

7. DISCUSSION AND CONCLUSIONS

In this study the mechanism for the upscale transfer of total energy from a localized heat source, such as a thunderstorm, has been investigated. An analysis of the energetics of a numerically simulated gravity wave, using the standard version of the RAMS, showed that it did not accomplish a transfer of total energy away from the thunderstorm. In fact an energy budget for the whole domain indicated that there had been no increase in total energy and this was also demonstrated for a prescribed heat source. Some bulk-energy constraints for the anelastic equations were formulated which showed that for a hydrostatic semi-infinite atmosphere with constant buoyancy frequency there was no net change in internal and gravitational potential energies when a heat source was prescribed. Furthermore, the kinetic energy produced was only a small fraction of the heat released.

It was hypothesized that the missing energy might be attributable to the filtering of thermal compression waves from the system of equations. An analytic solution for a thermally forced one-dimensional compression wave was obtained. It was shown that for this system of equations the increase in internal energy is exactly equal to the heat input. The wave energy associated with thermal compression waves is small compared with the heat input. However, thermal compression waves effectively transfer total energy. The perturbation of the internal-energy field produced by the heat source propagates at the speed of sound.
An experiment was carried out with a fully compressible numerical model to test the hypothesis that thermal compression waves transfer total energy in a more general system that also allows for the existence of gravity waves. After the heat source was turned off the compression waves separated from the slower-moving gravity waves and were clearly responsible for a transfer of total energy. This result suggests, at least for the circumstances investigated in this paper, that an approximate expression for total energy, obeying an equation of flux conservative form, cannot be obtained for systems that eliminate thermal compression waves.

This conclusion would appear to include the sound-proof systems of equations discussed by Lipps and Hemler (1982), Lipps (1990), and Durrant (1989). The expression for the total energy of the pseudo-incompressible system (Eq. (60) of Durrant 1989) is similar in form to the total energy for the fully compressible system. Yet the elimination of thermal compression waves from the pseudo-incompressible system indicates that this quantity cannot be transferred at the speed of sound and, therefore, in this regard it does not correspond to the total energy of the fully compressible system. Nevertheless, the pseudo-incompressible system should be able to reproduce accurately the slower-moving gravity waves that occur in the experiment, and these are regions having locally large amplitudes of perturbation total energy (Fig. 11(d)). In the absence of significant dissipative effects the perturbation total energy associated with the gravity waves is fairly well conserved. The total-energy equation for the pseudo-incompressible system (Eq. (61) of Durrant 1989) may correctly represent total-energy conservation within the gravity waves, while neglecting the large transfer of total energy associated with thermal compression waves. The pseudo-incompressible system has the advantage of giving more accurate solutions in stable regions than the widely used anelastic system discussed by Wilhelmson and Ogura (1972).

The compression waves discussed in this study would be difficult to observe. Studies of sound or infrasound waves focus on waves which are periodic. However, the waves discussed in this study that would be produced by thunderstorms would have a wave front followed by irregular pulses due to cellular convective activity. The perturbations would be very small in magnitude since in three dimensions the energy density would be expected to decrease rapidly with distance from the source.

The convective available potential energy is defined by:

\[
CAPE = \int_{LCL}^{EL} \frac{g}{\theta_0} \frac{(\theta_p - \theta_0)}{\theta_0} \, dz
\]

where LCL is the lifted condensation level, EL is the equilibrium level, \(\theta_0(z)\) the potential temperature of the environment and \(\theta_p\) the potential temperature of a parcel lifted along a moist adiabat. CAPE is considered a measure of the energy available to the storm, and its importance has been recognized in early studies of convection (Moncrieff and Green 1972; Moncrieff and Miller 1976). In these studies steady state is assumed and an energy equation is derived for a parcel lifted along a moist adiabat. The available-potential-energy release is balanced by a rate of change of kinetic energy and a pressure-work term. However, CAPE is not a measure of the heat released by an air parcel moving along a moist adiabat. The actual heat released is equivalent to the area between the moist adiabat and the dry adiabat that the parcel would have moved along in the absence of condensation. This large difference that exists between CAPE and the actual heat release is also evident from considering the lifting of a parcel from near the surface to the tropopause in an updraught, recognizing that most of the moisture condenses and falls out. The amount of moisture which condenses is typically ~10 g kg\(^{-1}\). Therefore,
for a unit mass of air, the heat release is \( Lq \sim 25000 \text{ J kg}^{-1} \), which is far larger than typical values of \( \text{CAPE} \sim 2000 \text{ J kg}^{-1} \). The results of this study provide an explanation for where all this energy goes.

A qualitative estimate demonstrates that the kinetic energy associated with gravity waves, which are generated by a heat source, is only a small fraction of the heat input. This is consistent with the observation that kinetic energy in the atmosphere is very small compared with total energy and with large-scale analyses of the generation of available potential energy. The heat input goes directly into increasing the internal energy and gravitational potential energy of the atmosphere, with only a small amount being converted into kinetic energy. However, the total energy does not simply increase within the immediate vicinity of the heat source, it is transferred at the speed of sound, so that the total energy increase is quickly distributed over a very large volume. The rapid vertical adjustments of the total energy in a column resulting from expansion and compression, in regard to the maintenance of a hydrostatic balance, has been discussed previously (see, for instance, Johnson 1970). It is hoped that the results of this study will help to clarify the role played by thermal compression waves, particularly in the horizontal transfer of total energy. Although the focus of this study has been on convective heat release, these results should also be applicable to sensible heating at the surface and radiational heating. Cooling will operate in a similar manner; total energy will decrease in a large volume which is expanding at the speed of sound.

If a certain amount of heat is released into a parcel of air, then the internal energy per unit mass of the air parcel will increase. However, because the air parcel expands, the internal energy per unit volume does not change. On a molecular scale, the mean energy of the molecules increases in the heated region, but their concentration per unit volume decreases, resulting in no net change of energy per unit volume.

Total energy is considered a fundamental quantity that is transferred poleward by the general circulation of the atmosphere, and has been the subject of numerous observational studies. The total energy flux in Eq. (6) is typically decomposed into the contributions by transient eddies, stationary eddies and mean meridional circulations in order to get a better understanding of the physical mechanisms involved in the fluxes (Oort and Peixóto 1983). The impression is that the large-scale advective motions in the atmosphere are directly responsible for the transfer of total energy. Certainly the advection of moisture, or latent heat, would be expected to result in a total-energy transfer. It also appears that the poleward flux of warm air and the equatorward flux of cold air within baroclinic eddies is considered to accomplish a sensible-energy transfer. In this article, the focus has been on the upscale transfer of total energy from a small-scale heat source, and it is not clear what the implications of these results are for the large-scale fluxes in Hadley circulations and baroclinic eddies. Nonetheless, it has been demonstrated that the quantity total energy can be effectively transferred at the speed of sound, without being associated with significant temperature changes. The results of this study strongly suggest that this mechanism for the transfer of total energy should be taken into consideration in physical interpretations of the atmospheric energy cycle.

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APPENDIX A

The thermodynamic equation used in the RAMS is given by:

\[ \frac{\partial}{\partial t} \theta_b = \text{ADV}(\theta_b) + \text{TURB}(\theta_b) + S(\theta_b) \]  

(A.1)

where \( \theta_b \) is the ice-liquid water potential temperature (Tripoli and Cotton 1981). For this study, \( \theta = \theta_b \), ADV and TURB are advective and turbulence operators, respectively (see Tripoli and Cotton 1982). The source term, \( S \), is related to the heating rate per unit volume, \( Q_v \), by,

\[ S = Q_v \theta / \rho \sigma T. \]  

(A.2)

APPENDIX B

The solution for the case where the heating is turned off at \( t = t^* \) is

\[ p'(x, t) = \frac{\gamma p_0 Q_0 a}{c_p T_0 c^2} \left\{ \begin{array}{l}
\frac{\arctan \left( \frac{c(t^* + t') + x}{a} \right)}{2} + \frac{\arctan \left( \frac{c(t^* + t') - x}{a} \right)}{2} - \\
\arctan \left( \frac{x + ct}{a} \right) + \arctan \left( \frac{x - ct}{a} \right)
\end{array} \right\} \]  

(B.1)

\[ u(x, t) = -\frac{\gamma p_0 Q_0 a}{\rho_0 \sigma T_0 c^2} \left\{ \begin{array}{l}
\frac{\arctan \left( \frac{c(t^* + t') + x}{a} \right)}{2} - \frac{\arctan \left( \frac{c(t^* + t') - x}{a} \right)}{2} - \\
\arctan \left( \frac{x + ct}{a} \right) - \arctan \left( \frac{x - ct}{a} \right)
\end{array} \right\} \]  

(B.2)

\[ \rho'(x, t) = \frac{\gamma p_0 Q_0 a}{c_p T_0 c^3} \left\{ \begin{array}{l}
\frac{\arctan \left( \frac{c(t^* + t') + x}{a} \right)}{2} + \frac{\arctan \left( \frac{c(t^* + t') - x}{a} \right)}{2} - \\
\arctan \left( \frac{x + ct}{a} \right) + \arctan \left( \frac{x - ct}{a} \right) - \frac{\gamma p_0 Q_0 T}{c_p T_0 c^2}
\end{array} \right\} \]  

(B.3)

where \( t' \) is measured from the instant the heating is turned off.

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