Chaos Theory and Its Applications to the Atmosphere

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Abstract

A brief overview of chaos theory is presented, including bifurcations, routes to turbulence, and methods for characterizing chaos. The paper divides chaos applications in atmospheric sciences into three categories: new ideas and insights inspired by chaos, analysis of observational data, and analysis of output from numerical models. Based on the review of chaos theory and the classification of chaos applications, suggestions for future work are given.

1. Introduction

Nonlinear phenomena occur in nature in a wide range of apparently different contexts, such as hydrodynamic turbulence, chemical kinetics, electronics, ecology, and biology; yet they often display common features or can be understood using similar concepts, permitting a unification of their studies. The similarity of complicated behaviors is not a superficial similarity at the descriptive level; instead, it concerns experimental and theoretical details. This similarity results from the modern theory of nonlinear dynamical systems, which describes the emergence of chaos out of order and the presence of order within chaos. This includes such features as solitons, coherent structures, and pattern formation, as well as chaos theory, which makes use of fractal dimensions, Lyapunov exponents, the Kolmogorov–Sinai entropy, and other quantities to characterize chaos. In this paper, only chaos theory will be reviewed.

Many of the publications in the past few years concerning chaos applications to the atmosphere have concentrated on the evaluation of fractal dimensions from observational data. The existence of low-dimensional climate and weather attractors is a highly debated subject. A more difficult task is to find concrete examples that show the significance of such computations. However, it needs to be emphasized that these computations are just a small portion of the applications of chaos theory to the atmosphere.

The purpose of this paper is to give a brief overview of chaos theory and to discuss its applications to the atmosphere. Instead of discussing any concepts or specific subjects in detail, we attempt to give an overall picture of the field. We usually cite the original references and more recent review papers so that interested readers can easily find more detail concerning particular fields of study and can find places to start their own research in this field. Chaos theory is reviewed in section 2. Applications of chaos theory to the atmosphere are discussed in section 3. Conclusions and suggestions for future research are given in section 4.

2. Chaos theory

a. Background

Li and Yorke (1975) seem to be the first to introduce the word *chaos* into the mathematical literature to denote the apparently random output of certain mappings, although the use of the word *chaos* in physics dates back to L. Boltzmann in the nineteenth century in another context unrelated to its present usage. However, there is still no universally accepted definition of the word *chaos*. Usually, chaos (deterministic chaos) refers to irregular, unpredictable behavior in deterministic, dissipative, and nonlinear dynamical systems. It should be emphasized that chaos cannot be equated simply with disorder, and it is more appropriate to consider chaos as a kind of order without periodicity. It was demonstrated in Lorenz (1963) that the sensitive dependence on initial conditions of a nonlinear system is related to the aperiodic behavior of the system.

By dynamical system we mean any system, whatever its nature, that can be described mathematically by differential equations or iterative mappings. Sometimes, we also include systems where the exact present state only approximately determines a near-future state: this extended definition of a dynamical system admits many real physical systems (such as the atmosphere plus its ocean and terrestrial boundaries), whose behavior commonly involves at least some randomness or uncertainty (Lorenz 1990). In a dissipative dynamical system, a vast number of modes die out due to dissipation, and the asymptotic state of the system can be described within a subspace of a much lower dimension, called the attractor. Chaos can also

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occur in Hamiltonian or conservative systems, but this subject requires special methods due to the absence of an attractor. Chaos may also occur in quantum systems where the number of degrees of freedom is larger than the number of independent commuting operators (Berman 1990). Another behavior, called intransitivity, which is different from chaos but related to it, also occurs in some, but not all, dynamical systems. An intransitive system is one with a positive probability of acquiring any one of several sets of infinite-term (or very long-term) properties (Lorenz 1990). In this paper, only deterministic chaos in dissipative nonlinear dynamical systems will be discussed.

There were classical physicists and mathematicians, even in the previous century, who had thought about nonlinear dynamical systems. Hadamard (1898) first observed the sensitivity of solutions to initial conditions at the end of the last century in a rather special system called geodesic flow. Subsequently, Poincaré (1908) discussed sensitivity to initial conditions and unpredictability at the level of scientific philosophy. (Poincaré even went on to discuss the problem of weather predictability!) However, their ideas seem to have been forgotten until Lorenz (1963) rediscovered them independently more than half a century later in his elegant paper entitled "Deterministic Nonperiodic Flow." The primary reason for this long hiatus is that chaos defies direct analytic treatment, making numerical computation essential. Therefore, Lorenz is generally regarded as the first to discover the irregular behavior and to analyze it quantitatively in completely deterministic, dissipative systems. He used a set of three equations, drawn from the spectral equation set of Saltzman (1962), to model the nonlinear evolution of the Rayleigh–Bénard instability (i.e., the instability that results when a fluid layer subjected to gravity is heated sufficiently from below). This model is also equivalent to a low-order quasigeostrophic model derived from shallow-water equations (Lorenz 1980) or derived from a two-level baroclinic model (Klein and Pedlosky 1992). By a careful analysis of the numerical solutions combined with analytical reasoning, Lorenz was able to deduce that the solution of his equations is eventually trapped in a region of the system's phase space that has a very intricate (strange) geometric structure, and this solution is very sensitive to the initial conditions.

Eight years later, Ruelle and Takens (1971), making use of then-recent developments in mathematics, proposed a possible mechanism for the transition from laminar flow to turbulence. Ruelle and Takens were the first to introduce the concept of a strange attractor, which is topologically different from other attractors, such as point attractors, which lead to steady-state solutions, limit cycles, which lead to periodic solutions, and tori, which lead to quasiperiodic solutions. The name *strange attractor* refers to its unusual properties, the most significant being sensitivity to initial conditions: two initially close trajectories on the attractor eventually diverge from one another. Strange attractors are finite-dimensional, and, in some sense, they correspond to exciting only a finite number of degrees of freedom; yet they have an infinite number of basic frequencies (Ruelle 1990). These two independent, pioneering papers triggered an upsurge of interest among researchers in different fields in an attempt to gain new insights. Since then, especially since 1975, publications related to chaos have grown extremely rapidly, and it is not the purpose of this paper to review all of this progress. Many of the historical papers on chaos were assembled into a single reference volume by Hao (1984). A comprehensive treatment of chaos theory with a readable account of many aspects of the subject may be found in Bergé et al. (1984). A nontechnical discussion of chaos is given in Gleick (1987). The concepts of chaos, fractal dimensions, and strange attractors, and their implications in meteorology, are presented in Tsonis and Elsner (1989) in a very readable way. Some more recent references can be found in, for example, Campbell (1990) and Marek and Schreiber (1991). However, most of the progress in this field so far may be roughly divided into two different categories: one involves bifurcations and routes to turbulence, and the other consists of quantitative means to recognize, characterize, and classify attractors. Here, we give only a brief review: usually only the original references and a few review papers will be cited.

b. Bifurcations and routes to turbulence

Ruelle and Takens (1971) showed that the Landau–Hopf route to turbulence (Landau 1944; Hopf 1948) is unlikely to occur in nature, and they instead proposed a route based on four consecutive bifurcations: fixed point $\rightarrow$ limit cycle $\rightarrow$ 2-torus $\rightarrow$ 3-torus $\rightarrow$ strange attractor (turbulence). A few years later, in collaboration with Newhouse, they reduced this scheme to fixed point $\rightarrow$ limit cycle $\rightarrow$ 2-torus $\rightarrow$ strange attractor. In other words, quasiperiodic motion on a 2-torus (i.e., with two incommensurate frequencies) may lose stability and give birth to turbulence directly (Newhouse et al. 1978). This result also implies that, usually, there are only four types of stable attractors (fixed point, limit cycle, 2-torus, and strange attractor) in a nonlinear dynamical system. This route, called the Ruelle–Takens route to turbulence, is generic and is known to occur in many mathematical models and laboratory experiments, but it remains less well understood theoretically than the other two routes mentioned below.

In an excellent review, May (1976) called attention to the very complicated dynamics, including period doubling and chaos, that can occur in very simple iterative mappings. Subsequently, Feigenbaum (1978, 1979a) discovered scaling properties and universal constants for one-dimensional mappings, and he introduced renormalization-group theory into this field. In addition, Feigenbaum proposed another route to turbulence, which is now called the Feigenbaum or period-doubling route to turbulence: a period-doubling bifurcation cascade with periods $p = 2^n$ ($n = 0, 1, 2, \ldots$) that quickly converges to an aperiodic orbit as $n \to \infty$. This scenario is extremely well tested in both numerical and physical systems. The period doublings have been observed in experiments such as Rayleigh–Bénard convection.

The third route, called the Pomeau–Manneville route to turbulence, is through intermittency (Pomeau and Manneville 1980; Manneville and Pomeau 1980). In the context of chaos, the term *intermittency* refers to random alternations of chaotic and regular behavior in time without involving any spatial degrees of freedom. This is slightly different from the original meaning of intermittency in the hydrodynamic theory of turbulence, which denotes random bursts of turbulent motion on the background of laminar flow. This scenario and the Feigenbaum scenario are, in fact, twin phenomena (Hao 1984), but the mathematical status of this third route is somewhat less satisfactory than those of the other two routes mentioned above, because, within its parameter regime, there is an infinite number of (very long) stable periods, and because there is no clear understanding of when the turbulent regime is reached or what is the exact nature of this turbulence (Eckmann 1981). Intermittent transitions to turbulence have been seen in many physical experiments.

Although we are facing a situation of "all routes lead to turbulence," the above three routes are the most thoroughly studied. The Ruelle–Takens route is related to Hopf bifurcations, where a pair of complex eigenvalues of the linearized map cross the unit circle; the Feigenbaum route is associated with pitchfork bifurcations, where an eigenvalue crosses the unit circle at −1; and the Pomeau–Manneville route is associated with saddle-node bifurcations, where an eigenvalue crosses the unit circle at +1. A more detailed discussion of these routes is given in a review paper by Eckmann (1981). Related to bifurcations are *crises* of chaotic attractors, which are abrupt changes of strange attractors themselves at certain parameter values; this subject is discussed in Grebogi et al. (1982) and Sommerer et al. (1991).

The reason for such intensive studies of the routes to turbulence is the belief that the key to understanding turbulence may be hidden in its onset mechanism, as pointed out by Landau (1944). Turbulence has been a long-standing problem in physics. It is no longer a specific problem in hydrodynamics; instead, it has become a general concept, relevant to many fields of science (e.g., solid-state turbulence, chemical turbulence, acoustic turbulence, and optical turbulence). On the other hand, it needs to be emphasized that chaos, at least for the time being, concerns mainly irregular behaviors in the temporal evolution, and is related only to the onset mechanism of turbulence—i.e., to weak turbulence. In contrast, fully developed turbulence involves both temporal and spatial irregularities. Patterns and spatiotemporal chaos were discussed in Campbell (1990). A recently developed technique called the wavelet transformation (Meneveau 1991, and references therein) provides a new tool for studying spatial intermittency and spatiotemporal nonlinear variations.

c. Characterization of chaos

A simple way to characterize attractors is via power-spectrum analysis, which is often used to qualitatively distinguish quasi-periodic or chaotic behavior from periodic structure and to identify different periods embedded in a chaotic signal. Chaos is characterized by the presence of broadband noise in the power spectrum. For example, Feigenbaum (1979b) used power-spectrum analysis to study the onset spectrum of turbulence. Poincaré maps (e.g., Berge et al. 1984) are sometimes also helpful in analyzing chaos. More sophisticated tools include Lyapunov exponents and various definitions of dimension.

There are three distinct intuitive notions of dimension: the topological dimension, . . . related to the number of directions in a space; the fractal (or Hausdorff) dimension, . . . related to the capacity of a space; and the information dimension, . . . related to the measurements made in a space.

1) Dimensions of attractors

In dissipative systems, the dimension D of an attractor is lower than the dimension k of the original phase space (which will be defined in section 2c), since some modes will damp out due to dissipation. There are three distinct intuitive notions of dimension: the topological dimension, which is related to the number of directions in a space (Hurewicz and Wallman 1948); the fractal (or Hausdorff) dimension, which is related to the capacity of a space (Mandelbrot 1983); and the information dimension, which is related to the measurements made in a space (Farmer 1982). The topological and fractal dimensions require only a metric (i.e., distance) for their definitions, whereas the information dimension needs both a metric and a probability measure for its definition (Farmer 1982). For simple attractors such as fixed points, limit cycles, or two tori, the separate notions of dimension lead to the same integer value. However, for chaotic (strange) attractors, these dimensions may be different, and the latter two may be noninteger.

In practice, the fractal dimension is the most widely used. It is one of the commonly used measures of the "strangeness" of attractors and is related to the number of degrees of freedom. It provides a lower bound on the number of dependent variables needed to describe the time evolution of the dynamical system, and, for a simple system, the Whitney embedding theorem (Takens 1981) provides an upper bound. However, for complex systems such as the atmosphere, the conditions of the Whitney embedding theorem may not be satisfied, and this upper bound may not be valid. The analysis of the correlation dimension (Grassberger and Procaccia 1983a), which is used as an estimate of the fractal dimension, also provides useful information about the variability of the system, information that goes beyond the bounds of traditional linear statistics. Such analyses reveal the extent to which the actual variations are concentrated on a limited subset of the space of all possible variations (Pierrehumbert 1990).

The fractal dimension mentioned above is sometimes called monofractal, and a multifractal spectrum is more appropriate for describing many systems. Multifractal measures are fundamentally characterized not by a single dimension value, but by a dimension function, which is simply related to a probability distribution (e.g., Lovejoy and Schertzer 1990, and references therein). This dimension function D_q provides an infinite number of different (and relevant) generalized dimensions (multifractal dimensions) that characterize an attractor. It is shown in Hentschel and Procaccia (1983) that D_0 is the Hausdorff dimension, D_1 is the information dimension, and D_2 is the correlation dimension (Grassberger and Procaccia 1983a). Multifractal measures can also be characterized by a spectrum $F(\alpha)$ of singularities, and a formal relationship between D_q and $F(\alpha)$ is derived by Halsey et al. (1986). Further refinements are discussed in, for example, Lovejoy and Schertzer (1990, and references therein). In the case of multifractality in fully developed turbulence, She and Orszag (1991) offered an explanation of the physics behind the multiple exponents—namely, local distortion of turbulent structures, which modifies the behavior of higher-order moments differentially.

The fractals described above are often called thin fractals to distinguish them from fat fractals, which are sets with fractal structure, but a nonzero measure (Eykholt and Umberger 1988). These latter sets are characterized by a fat-fractal exponent, rather than a dimension. They allow quantitative analyses of sensitivity to parameters, final-state sensitivity, and quantum chaos. Because they have finite measure, fat fractals must have the same (integer) dimension as the underlying space, and, as a result, their dimension is insensitive to their fractal structure.

2) Lyapunov exponents of strange attractors

The complexity of a strange attractor cannot be characterized merely by its dimension: such an attractor must be stretched and folded in some directions as well. These more subtle features can be described by Lyapunov exponents.

Lyapunov exponents are the average rates of exponential divergence or convergence of nearby orbits. The spectrum of Lyapunov exponents provides a quantitative measure of the sensitivity of a nonlinear system to initial conditions (i.e., the divergence of neighboring trajectories exponentially in time), and it is the most useful dynamical diagnostic for chaotic systems. Lyapunov exponents are independent of initial conditions on any orbit (Eckmann and Ruelle 1985). Any system containing at least one positive Lyapunov exponent is defined to be chaotic, with the magnitudes of the positive exponents determining the time scale for predictability. There are as many Lyapunov exponents as the dimension of the phase space (Guckenheimer and Holmes 1983), and, for a system of coupled ordinary differential equations, one of these exponents is necessarily equal to zero, meaning that the change in the relative separation of initially close states on the same trajectory is slower than exponential. The negative exponents express the exponential approach of the initial states to the attractor. In any well-behaved dissipative dynamical system, the sum of all of the Lyapunov exponents must be strictly negative (Guckenheimer and Holmes 1983).

Related to the Lyapunov exponents is the Kolmogorov–Sinai entropy (Kolmogorov 1958; Sinai 1959), whose inverse gives an estimate of the mean *e*-folding time of the initial growth of small errors. If the Lyapunov-exponent spectrum can be determined, then the Kolmogorov–Sinai entropy is bounded by the sum of all of the positive exponents (Eckmann and Ruelle 1985), and the fractal dimension may be estimated from the Kaplan–Yorke conjecture (Fredrickson et al. 1983), which will be explained later. Furthermore, the Lyapunov-exponent spectrum can be used to constrain the choice of mapping parameters in a prediction problem (Abarbanel et al. 1990). Note that Lyapunov exponents are the time averages of the local (temporal) rates of divergence. Higher moments of these local rates are also helpful in understanding the fine structure of the attractor.

For low-dimensional systems, even without a knowledge of the exact values of the Lyapunov exponents (which are important, as discussed in the previous paragraph), a knowledge of their signs alone can provide a qualitative characterization of the attractor. For instance, in a three-dimensional phase space, a set of three negative Lyapunov exponents (–, –, –) corresponds to a fixed point, (0, –, –) to a limit cycle, (0, 0, –) to a 2-torus, and (+, 0, –) to a strange attractor.

3) Computation of chaotic quantities

The computation of dimensions and Lyapunov exponents requires the technique of phase-space reconstruction. For a system of known first-order ordinary differential equations (ODEs) or difference equations, the set of all dependent variables constitutes a phase space—that is, a Euclidean space whose coordinates are these variables. Each point in this phase space represents a possible instantaneous state of the system. A solution of the governing equations is represented by a particle traveling along a trajectory in this phase space. If the solution is chaotic, then this trajectory is a strange attractor. Note that a single point in phase space determines this entire future trajectory, since such a point represents a complete set of initial conditions for the governing equations. In particular, this means that distinct phase-space trajectories can never cross, which is not the case in the three-dimensional physical space. Basically, these are the reasons that phase space is the natural setting for studying the time evolution of physical systems.

For a system with known partial differential equations (PDEs), the system can usually be studied by discretizing the PDEs, and the set of all dependent variables at all grid points constitutes a phase space, which is an approximation to the original infinite-dimensional phase space. For such a system (e.g., the atmosphere), an additional difficulty is that the initial values of many of the variables may be unknown. However, a time series of a single variable of a complex system may be available, and this allows the attractor of the system to be reconstructed. The physics behind such a reconstruction is that a nonlinear system is characterized by self-interaction, so that a time series of a single variable can carry the information about the dynamics of the entire multivariable system.

When dealing with a time series, the attractor can be reconstructed by any of three methods: the method of delay (Takens 1981), the method of derivatives (Packard et al. 1980), and the singular-value decomposition method (Broomhead and King 1986), which is also referred to as principal-component analysis or empirical–orthogonal function analysis in the literature of atmospheric science. These three methods can sometimes be improved by linear filtering, but such filtering must be done carefully, since it may increase the dimension of the time series (Badii et al. 1988). At this time, there is no general agreement about which method is best. However, for a short time series of low precision, the simple method of using delay coordinates is widely used, and it has been shown to work reasonably well in many situations (Zeng et al. 1992a,b, and references therein).

When delay coordinates are used with an infinite amount of noise-free data, the time delay τ can be chosen almost arbitrarily (Takens 1981). However, when only a limited amount of noisy data is available, the quality of the analysis depends on the value chosen for τ. The appropriate choice of this delay time depends sensitively on the attractor under study (Frank et al. 1990). Different methods have been suggested for obtaining τ, including the space-filling method (Fraedrich 1986), the method of computing the autocorrelation function, and the mutual-information method (Fraser and Swinney 1986; Fraser 1989). The mutual information measures general dependence, rather than linear dependence. For long time series, the mutual-information method may be the most comprehensive method; however, for many systems, it does not provide substantially different delay times than the other methods. Furthermore, in practice, when the data size is limited, it may not be possible to compute the mutual information accurately. In contrast, the autocorrelation function can be computed from small datasets, and the value of τ computed using this method does not differ substantially from the value based on higher-order autocorrelations. Therefore, for short time series of low precision, the simple autocorrelation method, combined with the space-filling method, may be used to determine the delay time τ for use in defining the delay coordinates.

It is usually difficult and impractical to compute various dimensions directly. Among the different procedures that have been developed to estimate fractal dimensions are the nearest-neighbor method (Badii and Politi 1985), the correlation-integral method (Grassberger and Procaccia 1983a), and the singular-system method (Broomhead and King 1986). Some information about the quality of the results obtained with the different methods has been reported (Holzfuss and Mayer-Kress 1986). In practice, the correlation-integral method is the most widely used, and the correlation dimension ν_s given by Grassberger and Procaccia provides a rigorous lower bound to both the information dimension and the fractal dimension (or Hausdorff dimension), and all three are generally close in value.

When the Lyapunov-exponent spectrum λ_i ($i = 1,2, ..., n$) with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ can be obtained, the Lyapunov dimension D_L is given by the Kaplan–Yorke conjecture (Fredrickson et al. 1983):

$$D_L = l + |\lambda_{l+1}|^{-1} \sum_{j=1}^{l} \lambda_j,$$

where l is defined by

$$\sum_{j=1}^{l} \lambda_j \geq 0 > \sum_{j=1}^{l+1} \lambda_j.$$

This dimension is related to the information dimension (Fredrickson et al. 1983), and its value is usually close to that of the correlation dimension. For systems of known ODEs or mappings, the Lyapunov-exponent spectrum can be obtained, and this provides a simple method for estimating the dimension. However, for a very complex system (e.g., the atmosphere), we cannot determine all of the Lyapunov exponents accurately from observational data, so this relation cannot be used.

The Lyapunov exponents can be computed relatively easily for simple known model systems (Shimada and Nagashima 1979). However, in many real-world situations, all that is available is a time series of experimental data, and it is much more difficult to extract the Lyapunov exponents from such a series. Only in the past few years have such methods been proposed. These methods differ only with respect to the orthonormalization method (Gram–Schmidt orthonormalization or Householder QR decomposition), the local mapping method (linear or higher-order polynomial), and some technical details. Early methods were based on the linearized mapping and either Gram–Schmidt orthonormalization (Wolf et al. 1985; Sano and Sawada 1985) or QR decomposition (Eckmann et al. 1986). Most of the later methods were based on Eckmann et al. (1986), replacing the linearized mapping with higher-order Taylor series (Briggs 1990; Bryant et al. 1990; Brown et al. 1991), and/or using the singular-value decomposition technique to determine the embedding dimension (Stoop and Parisi 1991). Based on the work of Sano and Sawada (1985) and Eckmann et al. (1986), we have recently proposed a practical method for estimating the Lyapunov-exponent spectrum from short time series of low precision (Zeng et al. 1991). A detailed discussion of different algorithms and related techniques in Zeng et al. (1992b) has shown that this algorithm works for short time series of low precision, while most of the above methods are useful only when the time series are long and/or have high precision.

When the correlation dimension can be estimated reliably, a relatively easy procedure for computing the Kolmogorov–Sinai entropy (Kolmogorov 1958; Sinai 1959) was developed by Grassberger and Procaccia (1983b). When the Lyapunov exponents can be obtained, the Kolmogorov–Sinai entropy is bounded by the sum of all positive exponents. The finite and positive Kolmogorov–Sinai entropy is a basic quantity characterizing chaotic behavior, and its inverse gives an estimate of the mean predictability time of the system.

3. Applications of chaos theory

We divide chaos applications (e.g., in the atmosphere) into three broad categories: new ideas and insights inspired by chaos, analysis of observational data, and analysis of output from numerical models.

a. New ideas and insights

The first category consists of new ideas and physical insights inspired by chaos. Just as relativity eliminated the Newtonian illusion of absolute space and time, and as quantum theory eliminated the Newtonian and Einsteinian dream of a controllable measurement process, chaos eliminates the Laplacian fantasy of long-term deterministic predictability. Because of chaos, it is realized that even simple systems may give rise to and, hence, be used as models for complex behavior. Conversely, complex systems may give rise to simple behavior (e.g., coherent structures), which may be predicted for a period of time within the predictability limits. Finally, and most important, the laws of scaling and complexity hold universally, caring not at all about the details of the system. Chaos leads to the unification of order and disorder, and to the unification of deterministic and stochastic descriptions (Lorenz 1987). Chaos also acts like a bridge between research in traditionally unrelated fields: in fact, the International Federation of Nonlinear Analysts (IFNA) was established in August 1991, with members in a variety of scholarly disciplines. Much of Lorenz's research presents excellent examples of how physical insights and new ideas can be gained by studying chaos (e.g., Lorenz 1991a, and references therein).

Although chaos places a fundamental limitation on long-term prediction, it suggests a possibility for short-term prediction: random-looking data may contain simple deterministic relationships, involving only a few irreducible degrees of freedom. Therefore, chaos theory has also been used for prediction problems, especially when there is a lack of proper initial data for use in a numerical prediction model, or when a good model is lacking (Abarbanel et al. 1990; Elsner and Tsonis 1992; and references therein). Elsner and Tsonis (1992) also reviewed the prediction problems involved with neural networks. This is currently a very active research area.

Since realistic atmospheric models, with or without stochastic terms, ordinarily have general solutions that are aperiodic, they tend to yield similar results in either case (Lorenz 1987). This is the implicit assumption behind numerical weather predictions based on deterministic approaches. Note, however, that this does not mean that the atmosphere actually is deterministic, since, for example, external forcing such as solar output may need to be represented stochastically.

As mentioned in section 2c, Lyapunov exponents are the time averages of the local (temporal) rates of divergence, and higher moments of these local rates are helpful in understanding the fine structure of the attractor. Nese (1989) investigated both the temporal and phase-spatial variations in predictability of the Lorenz model (Lorenz 1963). Nicolis (1992) developed a probabilistic approach accounting for the variability of error growth in the atmosphere and applied it to a low-order model (Lorenz 1990). Nicolis found a wide dispersion around the mean, showing the inadequacy of a description limited to averaged properties only. Since Lyapunov exponents and the Kolmogorov–Sinai entropy are related to the predictability of a system, the longevity of the enhanced predictability periods often observed in the atmosphere can be quantified by computing the higher statistical moments of error growth rates. In fact, it is argued in Benzi and Carnevale (1989) that the ratio of the average growth rate to the most probable one is a measure of enhanced predictability. It is also shown that the atmospheric predictability depends on the scales of the flows, on the flow regimes, on the time average (in contrast to instantaneous values), and on measures of forecast skills (Schubert et al. 1992; Zeng 1992; and references therein).

Lorenz (1969) hypothesized that the growth of small initial errors in the atmosphere follows a quadratic law, and this hypothesis has been verified for several numerical models (Lorenz 1982a; Dalcher and Kalnay 1987; Trevisan et al. 1992). However, deviations from this hypothesis are also observed (Chen 1989; Schubert and Suarez 1989). Nicolis and Nicolis (1991) showed that the time evolution of initial errors in unstable systems follows three different stages: the initial exponential stage, the quasi-linear stage, and the saturation stage. The first stage reflects local (linearized) properties, and the last two stages depend on global properties.

Chaos theory in dissipative dynamical systems (e.g., the atmosphere) has shown that, no matter how large the original phase space, the final state may be described by motions in subspaces (attractors) of much lower dimension. Low-order models obtained by truncating atmospheric governing equations are often used as a first step toward studying atmospheric processes. After all, chaos was found by Lorenz (1963) in a truncated model. In general, low-order models are often capable of representing atmospheric processes in a qualitatively correct manner, and a general procedure for constructing them was discussed by Lorenz (1982b). The influence of truncation on the results in Lorenz (1990) was discussed by Wiin-Nielsen (1992). Charney and DeVore (1979) were the first to study the multiple equilibrium states in the atmosphere in a truncated barotropic model. The influence of truncation on the existence of multiple equilibria was studied by several researchers (Tung and Rosenthal 1985; Cehelsky and Tung 1987; and references therein). The influence of the level of dissipation and the form of the dissipation mechanism on the dynamical behaviors (including the chaotic behavior) of unstable baroclinic waves was studied by several researchers (Klein and Pedlosky 1992, and references therein).

The proper description and modeling of turbulent energy dissipation processes are vital for the understanding of fully developed turbulence. Meneveau and Sreenivasan (1987) proposed a multifractal model of the energy-cascade process in the inertial range that fits remarkably well the entire spectrum of scaling exponents for the dissipative field in fully developed turbulence. Furthermore, it has been shown that multifractal dimensions for the physical phenomena that arise from cascade processes can be characterized by three universal parameters that represent the degree of multifractality, the sparseness of the average energy, and the degree of conservation of the analyzed field (Schertzer and Lovejoy 1991a, and references therein). Several methods have been proposed to estimate these universal multifractal exponents (Schmitt et al. 1992, and references therein).

Scale invariance is a symmetry principle in which the small and large scales are related by a scale-changing operation that depends only on the scale ratios; there is no characteristic size. However, it usually cannot be applied directly to geophysical systems (including the atmosphere) due to the high anisotropy of the system caused by differential stratification, rotation, and other more complex scale-changing operations. The need to deal with scale-invariant anisotropy has led to the development of a model of generalized scale invariance (Schertzer and Lovejoy 1991a, and references therein). This model unifies the large- and small-scale dynamics by using an anisotropic regime with a single scaling, rather than the conventional use of two distinct isotropic regimes: a two-dimensional regime for the large-scale dynamics and a three-dimensional regime for the small-scale dynamics. The scaling hypothesis has been confirmed by many empirical studies of geophysical phenomena (Lovejoy and Schertzer 1991, and references therein) and, recently, by more systematic analyses (Lovejoy et al. 1993).

Climate dynamics has been a highly active research area in recent years, partly due to the concern over possible significant global change caused by increased levels of greenhouse gases. Using a very low-order geostrophic baroclinic "general circulation" model, Lorenz (1990) demonstrated the reasonableness of his earlier proposition that the climate system is unlikely to be intransitive—that is, to admit two or more possible climates, any one of which, once established, will persist forever. Lorenz also showed that chaos and intransitivity can lead to interannual variability by means of nonlinear wave–mean flow interactions. Pielke and Zeng (1993) integrated this low-order model for about 1100 years and found that, when the seasonal cycle is included, chaos and intransitivity can lead not only to interannual variability, as in Lorenz (1990), but also to the long-term natural variability on decade and century time scales that is as large as what occurs between years or within a year.

Nicolis (1990) outlined a general algorithm for casting deterministic chaos into a Markovian process and illustrated the theory by some examples of interest in atmospheric and climate dynamics. Such a statistical approach can be used for the long-term predictions of complex systems undergoing chaotic dynamics, such as the climate system. Tsonis and Elsner (1990b) used a forced nonlinear oscillator with damping to study long-term climate dynamics, and found that the system exhibits multiple attractors, that jumps between the attractors may take place in the presence of noise and fractal basin boundaries, and that the residence time on each of the attractors is a random variable whose mean value is different for each attractor. They argued that such a mechanism may provide an explanation for rapid deglaciations and the fact that ice ages do not last as long as today's climate. Note that their results may not be contradictory to Lorenz's results mentioned above, because an ice-age effect was not included in Lorenz (1990).

By finding chaos in Daisyworld (Watson and Lovelock 1983), Zeng et al. (1990) and Flynn and Eykholt (1993) raise questions regarding the interpretation and validity of the Gaia hypothesis that the evolution of the biota and their environment are a single, tightly coupled process with the self-regulation of climate and chemistry as a fundamental property of such a connected nonlinear system (Lovelock and Margulis 1974; Lovelock 1989).

Using a conceptual model, Vallis (1986) explained most of the principal qualitative features of the El Niño–Southern Oscillation phenomenon (including the aperiodic occurrence of these events). Using two-dimensional atmospheric flow, Rinne and Jarvinen (1992) showed that, when the numerical approximations do not allow air parcels to deviate from each other sufficiently, errors in the numerical approximations can prevent the correct prediction of chaotic processes. They proposed that this explains the errors observed when forecasting blocking highs. Bretherton (1990) used the turbulence cascade model of Meneveau and Sreenivasan (1987) to study the entrainment process at stratocumulus cloud top and found spatial intermittency in this entrainment process (i.e., entrainment takes place predominantly in large eddies), which can be tested by future aircraft measurements. Gauthier (1992) showed that, due to different local-error growth rates, the same observational error can lead to very different accuracies when applying the adjoint method to four-dimensional data assimilation.

Partial introductions to chaos and its applications to the atmosphere have also been presented in Tsonis and Elsner (1989) and Yang (1991). More applications can be found in Schertzer and Lovejoy (1991b) and Sreenivasan (1991). However, the influence of the change from traditional viewpoints brought about by the study of chaos still needs time to be fully observed, just as was the case with relativity theory and quantum theory.

b. Analysis of observational data

The second category of chaos applications involves the analysis of observational data. Chaos theory offers a fresh way to deal with observational data, especially those data that might otherwise be ignored because they proved too erratic. Many chaotic studies in the field of atmospheric science have concentrated on computing quantities characterizing attractors, especially fractal dimensions, from observational data. Nicolis and Nicolis (1984) analyzed the time series of the isotope record of deep-sea cores and obtained a low dimensionality (between 3 and 4) for the climate system. Subsequently, Fraedrich (1986, 1987), Essex et al. (1987), and Keppenne and Nicolis (1989) analyzed daily-average data over eastern North America and western Europe, and have likewise concluded the existence of low-dimensional attractors. Also, theories of deterministic chaos and fractal structure have been applied to data in the atmospheric boundary layer (Tsonis and Elsner 1988), the pulse of storm rainfall (Sharifi et al. 1990), and some special atmospheric systems such as the Southern Oscillation (Hense 1987). Using entire global fields of data, rather than single-point time series, Pierrehumbert (1990) discussed the dimension of global atmospheric variability.

The existence of low-dimensional atmospheric attractors is currently a highly debated subject. It is widely accepted now that there are limitations to the Grassberger–Procaccia algorithm when the number of data is limited. Some doubts among researchers concerning strange attractors in the atmosphere were discussed in Pool (1989). Qualitative data requirements needed to accurately calculate the chaotic characteristics of a nonlinear system were discussed in Essex et al. (1987) and Tsonis and Elsner (1990a). Various researchers have given different quantitative criteria, among which the criterion of Ruelle (1990) is the least strict. In Zeng et al. (1992a), the time series of observational data (with different climatic signal/noise ratios) has a length that is comparable to or greater than those used in previous studies; however, a saturated fractal dimension ν_s still could not be obtained, and it could be claimed only that ν_s is well above 8. Using the quantitative arguments of Ruelle (1990) and Nerenberg and Essex (1990), it is shown in Zeng (1992) and Zeng et al. (1992a) that most, if not all, of the previous estimates of low-dimensional attractors are unreliable. Based on simple models, Lorenz (1991b) proposed that, if a low fractal dimension can be obtained from observational data, this may instead reflect the weak nonlinear interaction between the observed variable and the other variables in the atmosphere. This gives another possible reason for apparently finding low-dimensional attractors in the atmosphere.

The above evaluations of fractal dimensions were carried out in phase space. However, such analyses can also be carried out for trajectories in physical space. Fraedrich and Leslie (1989) and Fraedrich et al. (1990) analyzed the data of cyclone tracks in the tropics and the midlatitudes both in phase space and in physical space. They found that the predictability of cyclone tracks depends on their geographical locations and the flow regimes. Gifford (1991) analyzed the observational data of pollutant concentrations and obtained a fractal dimension of 1.6 to 1.7 for the trajectories of the pollutants. This can be useful for large-scale atmospheric diffusion modeling, which currently assumes a fractal dimension of 1.5 (i.e., ordinary Brownian motion). Using satellite data for clouds, Cahalan and Joseph (1989) studied the spatial structure of the atmospheric boundary layer and the clouds of the intertropical convergence zone by examining the change in the fractal dimension for various conditions (e.g., the cloud type). The relationship between fractal dimension and turbulent diffusion, as well as general fractal concepts, was reviewed in Ludwig (1989).

It should be emphasized that the estimation of fractal dimensions from observational data is just a small portion of the applications of chaos theory to the atmosphere. A more fundamental question is: Even if a low-dimensional atmospheric attractor does exist for a particular type of atmospheric flow, how do we actually construct a mathematical mapping from just a few parameters to the actual atmospheric flow patterns related to the dominant physical processes revealed by the analyzed data? In a recent study by Zeng and Pielke (1993), a low-dimensional atmospheric attractor was obtained for surface thermally induced atmospheric flow, and parameters or physical processes related to these low dimensions were discussed. A linear model can be used to provide the mathematical mapping from the physical processes to the atmospheric flow fields that capture the salient features of the system. This study also implies that a low fractal dimension does not necessarily mean that the system can be described by a few equations; more generally, a low dimension means that the system can be described by a mathematical mapping from a few key physical processes or parameters to the flow fields. As far as we know, this is the first study in which such a mathematical map is actually given.

As mentioned before, a multifractal analysis is more appropriate for describing geophysical systems. Lovejoy and Schertzer and their groups have applied multifractals and generalized scale invariance to the study of radiation processes, satellite imagery of clouds, radar precipitation data, turbulence data, and other types of data (Lovejoy et al. 1993, and references therein). The multifractal properties of energy dissipation derived from turbulence data have also been studied by many researchers (Aurell et al. 1992, and references therein). Aurell et al. also discussed the occurrence of spurious scaling due to finite Reynolds number effects in the computation of multifractal dimensions (Halsey et al. 1986).

Related to the estimation of fractal dimensions is the estimation of Lyapunov exponents from observational data (Keppenne and Nicolis 1989; Zeng et al. 1992a). The relationship between predictability based on computing Lyapunov exponents and that based on general circulation models and the relationship between local and global predictabilities were also discussed in Zeng et al. (1992a). In contrast to the estimation of fractal dimensions, there are still no extensive discussions concerning the qualitative and quantitative data requirements for the computation of Lyapunov exponents. One brief study is that of Eckmann and Ruelle (1992), in which they claim that the number of data points needed to estimate Lyapunov exponents is about the square of that needed to estimate the correlation dimension. Some uncertainties caused by the selection of the time delay were reported in Zeng et al. (1992a), but were not mentioned in a similar study by Keppenne and Nicolis (1989).

The analysis of observational data is not limited to the estimates of chaotic quantities. Ghil and Vautard (1991) used singular-spectrum analysis to analyze time series of global surface air temperature for the past 135 years, allowing a secular warming trend and a small number of oscillatory modes to be separated from the noise. They showed that the combined amplitude of the oscillatory components (i.e., the natural variability) could postpone incontrovertible detection of a possible greenhouse warming signal for one or two decades.

c. Analysis of output from numerical models

The third category of applications of chaos theory deals with output from numerical models such as general circulation and mesoscale models (Zeng and Pielke 1993). As mentioned in section 2c, when the number of model equations (at all grid points) is not too large (e.g., less than 30), and when the parameterization of physical processes is not too complex, the Lyapunov-exponent spectrum can be obtained directly (Shimada and Nagashima 1979), and the fractal dimension can be estimated using the Kaplan–Yorke conjecture (Fredrickson et al. 1983). In the case of truncated models (e.g., 11 equations in Nese et al. 1987, or 21 equations in Lorenz 1991b), such methods can be used efficiently to estimate Lyapunov exponents and the fractal dimension. However, further work is needed to determine under what conditions (e.g., the number of model equations and the complexity of the parameterizations) such direct computations are practical.

For atmospheric numerical models, such as general circulation models, which typically have 10^5 grid points, the direct application of the method (Shimada and Nagashima 1979) is impractical, and the computation of chaotic quantities is the same as in the case of observational data. However, more data can be obtained from model output, and the phase space can be constructed more conveniently by using variables at different (and appropriate; that is, not closely correlated) locations. This avoids the problem of selecting an appropriate time delay, and significantly decreases the computational cost of estimating fractal dimensions.

The chaotic analysis of model output also provides a new way to compare model results with observations. It must be true that an accurate numerical simulation should produce fractal dimensions and Lyapunov exponents similar to those obtained from observational data. Further work is needed in this area.

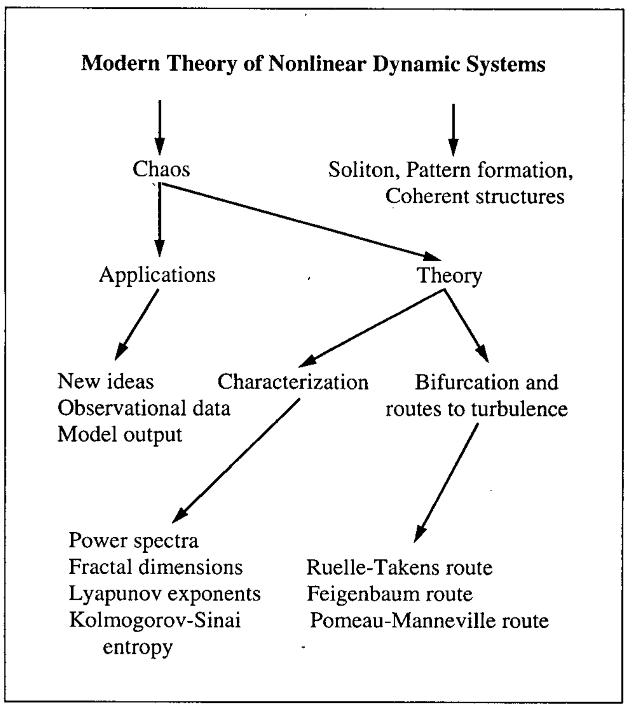

FIG. 1. Summary of chaos theory and its applications to the atmosphere. The computation of chaotic quantities is excluded.

4. Conclusions and suggestions for future research

Chaos theory has been reviewed, and its applications to the atmosphere have been divided into three categories. This can be summarized schematically in Fig. 1. This overview and classification also demonstrate that the current emphasis on the computation of fractal dimensions from observational data and the debate regarding the existence of low-dimensional attractors in the atmosphere represent just a small portion of applications of chaos theory to the atmosphere.

Since this work has evolved from our research on a variety of topics, a number of suggestions for future research have become apparent. Some of these suggestions relating to atmospheric research are mentioned here.

Although qualitative and quantitative data requirements have been established for the computation of monofractal dimensions, the corresponding requirements for the computation of the multifractal spectrum of dimensions or the Lyapunov-exponent spectrum, which is more important dynamically, have not yet been investigated. Further work is also needed to study the practical significance of computing fractal dimensions.

For high-dimensional complex systems (such as the atmosphere), the reconstructed embedding dimension may be smaller than the actual fractal dimension. This may be related to the slight sensitivity to the selected value of the time delay τ in the computation of the Lyapunov exponents from observational data, as reported in Zeng et al. (1992a). It is assumed in Zeng et al. (1992a) that at least the first few positive exponents can be reasonably estimated. Considering the practical significance of this assumption, further work is needed.

Predictability is closely related to the Lyapunov-exponent spectrum. We hypothesize that the predictability should be universal for certain types of dynamical systems. Stated in another way, our hypothesis is that the error growth after certain rescaling should be the same for families of nonlinear dynamical systems (e.g., those with quadratic nonlinearities). One way to test this hypothesis is to follow the milestone work of Feigenbaum (1978, 1979a), which led to the well-known Feigenbaum constants.

The estimation of Lyapunov exponents and predictability is usually related to the growth of small initial errors. Further work is needed to understand the quasi-linear and saturation stages of error growth.

Noise reduction (Kostelich and Yorke 1990; Elsner and Tsonis 1992; and references therein) is a very active research area in nonlinear science. For example, observational data can be processed by noise-reduction techniques before four-dimensional data assimilation.

Prediction based on observational data is also a very active research area in nonlinear science, because it provides a more stringent test of the underlying determinism in situations of a given complexity (Abarbanel et al. 1990; Elsner and Tsonis 1992). This is also helpful to atmospheric research (such as climate dynamics).

The computation of chaotic quantities and mutual information (Fraser and Swinney 1986; Fraser 1989) and the wavelet transformation (Meneveau 1991) provide new tools for model output analysis and for comparison of model results with observations. Further work is needed to apply these methods to different model outputs.

Note added in proof: Due, in part, to communications with several scientists, including Barry Saltzman, Brian F. Farrell, John E. Hart, and Roger Barry, we have become aware of several additional recent publications related to applications of chaos theory to the atmosphere. From laboratory experiments on oscillatory flows over topography in a rotating fluid, Pratte and Hart (1991) showed that periodic forcing can lead to both periodic and chaotic behaviors. Using a stochastic–dynamical global model of late Cenozoic climatic change, Saltzman and Maasch (1991, and references therein) demonstrated that both internal dynamics and earth-orbital (Milankovitch) forcing are responsible for the ice-age oscillations. Brindley et al. (1992) discussed the effect of an extra periodic or stochastic forcing on chaos and noisy periodicity in two simple forced ocean–atmosphere models. Butler and Farrell (1992) used three-dimensional optimal (nonmodal) perturbations in viscous shear flow to study the transition from laminar to turbulent flow. Selvam et al. (1992) discussed a nondeterministic cell dynamic system model for atmospheric flows. Tsonis (1992) discussed chaos theory and its applications. Waldrop (1992) presented a nontechnical discussion of complexity theory, which examines the systems that lie in the middle ground between the predictable and the chaotic. Islam et al. (1993) pointed out that, when a variable (e.g., precipitation) depends on physical constraints and thresholds, the chaotic analysis of this variable may lead to an underestimation of the correlation dimension of the underlying dynamical system. Using the simple model of Lorenz (1963), coupled with a linear oscillator, Palmer (1993) explored the physical basis for extended-range atmospheric prediction.

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