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# A Delayed Biophysical System for the Earth's Climate

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**Summary.** We consider further the Differential Daisyworld model of Watson and Lovelock that we have analyzed in a previous paper (De Gregorio et al., 1992). In this work we introduce a delay in the birthrate of the species. We consider three different models: the constant time lag model and the strong and the weak delay models. In the weak delay case no value of the delay changes the asymptotic stability of the stationary solutions. In the constant time lag and in the strong delay models, however, there exists a critical value of the delay, above which periodic solutions appear. These periodic solutions are numerically found to be globally attracting even for large delay when the linear approximation analysis is no longer valid. For both models, very regular behavior is obtained if the percentage coverage of the fertile ground of the Earth is much less than 1. As the percentage of the fertile ground increases, however, chaotic behavior is possible.

**Key words.** bioclimatic system behavior, constant time lag, strong and weak delay, Hopf bifurcation, asymptotic local and global stability

## 1. Introduction

The Earth's climate system is complex: many parameters play important roles, often through nonlinear feedbacks even in low-dimensional models (Budyko, 1969; Sellers, 1969; Lorenz, 1990). Furthermore, external parameters (Berger, 1977) and internal parameters (Ghil, 1976; Kallen et al., 1979) are important. A detailed analysis of the energetics of the climate system shows that it has a time-space-wide spectrum.

There is historical evidence that the albedo of the Earth has increased about 10 percent in the last 6000 years (Otterman, 1977; Neumann and Sigrist, 1978; Neumann and Parpola, 1987) because of changes in vegetation coverage. The climate system has been affected by these changes, and the centers of the anthropogenic activities have changed in time.

In this study we concentrate our attention on the feedback between a simple biosystem and the temperature of the Earth. More specifically, we study the effect of the time lag between the Earth's temperature and the response of the biosystem. We also explore extreme values of the delay in search of possible chaotic behavior of the bioclimatic system.

A simple dynamical model that considers the feedback of the solar luminosity and the environmental biosphere, and the connected consequences on the climate of the Earth, has been proposed by Watson and Lovelock (1983). This model has been recently studied both in a discrete map formulation (Zeng et al., 1990; Zeng et al., 1992; Flynn and Eykhold, 1992) and in a continuous formulation (De Gregorio et al., 1992). The outputs of the two formulations do not coincide, and there are many reasons for that.

Here we investigate a continuous system that has a clearer connection with the discrete model. In a discrete population model the $(n + 1)$th generation is a function of the $n$th generation. This means not only that the population responds in discrete jumps, but also that the current size of the system is a function of its size at a previous time. It is this last property that is included in the continuous delayed model; the evolution depends not only on the present values of the variables but also on the values at a previous time. We reconsider then the continuous model studied by De Gregorio and colleagues (1992) and modify it to include delays on the growth rate of the species with respect to external variations of luminosity. That is, we assume that the growth of the daisies depends not only on the present temperature over the daisies, but also on the temperatures at a fixed time before or, more generally, on all the previous temperatures.

This new formulation is, in principle, more realistic and gives a deeper understanding of the connection between solar luminosity, development of the environmental biosphere, and climate of the Earth, than the original ordinary differential model.

The introduction of the delay is motivated by the fact that biological species do not necessarily respond instantaneously to climate changes because of the gestation or germination time. In our model the birth coefficient depends not only on the present temperature, but also on the temperature previously experienced by the species during a determined interval of time. The delay can also simulate the effects of parameters not explicitly included in the model, such as melting ice with a resultant change of the albedo (Bhattarcharya et al., 1982).

## 2. The Continuous System

We assume the existence of two species of vegetation, referred to as daisies by Lovelock and Watson. To enhance the difference in albedo, we assume that the daisies are white and black. Their simple dynamical evolution is described, in a continuous formulation, by the two-dimensional system of ODEs

$$\begin{aligned} \dot{a}_b &= a_b\,[(p - z)\,\beta_b\,(T_b) - \gamma]\,, \\ \dot{a}_w &= a_w\,[(p - z)\,\beta_w\,(T_w) - \gamma]\,, \end{aligned} \tag{1}$$

where $a_{b,w}$ are the densities of the daisies, $z = a_b + a_w$, $p$ is the fraction of fertile ground, $\beta(T)$ is the coefficient of birth, assumed to depend on the local temperature $T$ over the daisies, and $\gamma$ is the coefficient of death, taken as a constant for simplicity. In this model, all bare ground, fertile or not, has the same albedo. (The introduction of a difference in albedo for different areas of the bare ground will not change the analysis and the qualitative results.) In this very idealized approach there are no oceans. The Earth is a zero-dimensional system that radiates according to Stephan Boltzman's law

$$(T_e + T_0)^4 = \frac{SL}{\sigma}(1 - A), \tag{2}$$

where $T_0 = 273$ K, $S$ is the heat flux, $L$ the solar luminosity, $A$ the mean albedo, and $\sigma$ the Stephan Boltzman constant. Also,

$$A = A_g a_g + A_b a_b + A_w a_w \tag{3}$$

with $A_g$, $A_b$, and $A_w$ denoting the albedos of the ground, and of black and white daisies, respectively, and the fractional coverage satisfying the condition

$$a_g + a_b + a_w = 1. \tag{4}$$

For the local temperature over the species we define

$$(T_{g,b,w} + T_0)^4 = \alpha\frac{SL}{\sigma}(A - A_{g,b,w}) + \frac{SL}{\sigma}(1 - A) \tag{5}$$

where $\alpha = 0$ corresponds to complete diffusion of the temperature (the local temperature coincides with the mean temperature) and $\alpha = 1$ corresponds to complete insulation

$$(T_i - T_0)^4 = \frac{SL}{\sigma}(1 - A_i), \qquad i = g, b, w, \tag{6}$$

(i.e., the local temperature does not depend on the fractional coverage of the other species).

The functions $\beta_i(T_i)$, $i = b, w$, are functions of the fractional coverage of the species. We define the values of the albedos of black and white daisies as symmetric with respect to the albedo of the bare ground and furthermore suppose that the $\beta$s are represented by the parabolic function

$$\beta(T) = 1 - \delta(T - \overline{T})^2$$

with $\overline{T} = 22.5°\text{C}$ and $\delta$ such that $\beta(5) = \beta(40) = 0$. Linearizing Stephan's law around $T = T_0 + \overline{T}$, we obtain the following system of ODEs of the fourth degree

$$\begin{aligned} \dot{a}_b &= a_b\left[(p - z)\left(\rho_b + \eta_b y - \nu y^2\right) - \gamma\right], \\ \dot{a}_w &= a_w\left[(p - z)\left(\rho_w + \eta_w y - \nu y^2\right) - \gamma\right], \end{aligned} \tag{7}$$

where $y = a_b - a_w$, and the coefficients $\rho_i$, $\eta_i$, $\nu$ are functions of the solar luminosity $L$ and of the diffusion coefficient $\alpha$; $\nu$ being the same for black and white daisies. This system has been studied in De Gregorio and colleagues (1992), and many of its properties have been determined analytically, properties then confirmed by the numerical simulations of the system (1), that is, without the approximations needed to obtain the analytical results. In this paper we add the delay for the growth-rate response to the external variations of solar luminosity. Details for the derivation of system (7) can be found in the previous paper.

## 3. The Delayed System

The introduction of the delay in system (7) poses two principal problems: first, which terms are to be considered to yield a delayed effect and, second, what form of delay response should be introduced?

Concerning the first problem, we assume that the coefficient of birth, $\beta(T)$, does not depend on the temperature at time $t$, but on the temperature at a previous time $t - \tau$, or even more appropriately, on all the temperatures preceding the time $t$. For a discrete time delay (also called a constant time lag), the explicit form assumed by system (7) is now

$$\frac{\dot{a}_i(t)}{a_i(t)} = (p - z(t))\,\beta_i\,(t - \tau) - \gamma, \qquad i = b, w, \tag{8}$$

which, written in an explicit form for each daisy, is the nontrivial system composed of the following equation

$$\begin{aligned} \frac{\dot{a}_b(t)}{a_b(t)} = \;& p\rho_b - \gamma - \rho_b\,[a_b(t) + a_w(t)] + \; p\eta_i\,[a_b(t - \tau)] - a_w(t - \tau) \\ & - \; \eta_b\,[a_b(t) + a_w(t)]\,[a_b(t - \tau) - a_w(t - \tau)] \\ & - \; p\nu\,[a_b(t - \tau) - a_w(t - \tau)]^2 \\ & + \; \nu\,[a_b(t) + a_w(t)]\,[a_b(t - \tau) - a_w(t - \tau)]^2 \end{aligned} \tag{9}$$

and a similar equation for $a_w(t)$. We know that the stationary solutions of a delayed system coincide with the stationary solutions of a nondelayed system. So we know already (De Gregorio et al., 1992) that there exists a stationary solution inside the triangle of the phase space

$$0 < a_b + a_w < p, \qquad a_b\,,\, a_w > 0, \tag{10}$$

which is locally attracting. It is given by

$$y_0 = \frac{\rho_b - \rho_w}{\eta_w - \eta_b} \tag{11}$$

and

$$z_0 = p - \frac{\gamma}{\rho_b + \eta_b y_0 - \nu y_0^2}. \tag{12}$$

Posing $a_b = a_b^0 + x$, $a_w = a_w^0 + y$, and $z_0 = a_b^0 + a_w^0$; the linear approximation of system (8) around the stationary solution is given by

$$\begin{aligned} \dot{x}(t) &= -\frac{a_b^0\gamma}{p - z_0}[x(t) + y(t)] + a_b^0(p - z_0)(\eta_b - 2\nu y_0)[x(t-\tau) - y(t-\tau)], \\ \dot{y}(t) &= -\frac{a_w^0\gamma}{p - z_0}[x(t) + y(t)] + a_w^0(p - z_0)(\eta_w - 2\nu y_0)[x(t-\tau) - y(t-\tau)]. \end{aligned} \tag{13}$$

The characteristic equation of this system is

$$\begin{aligned} &\lambda^2 + \lambda\left[\frac{z_0\gamma}{p - z_0} + e^{-\lambda\tau}(p - z_0)\left(2\nu y_0^2 + \eta_w a_w^0 - \eta_b a_b^0\right)\right] \\ &\quad + 2\gamma e^{-\lambda\tau} a_b^0 a_w^0 (\eta_w - \eta_b) = 0. \end{aligned} \tag{14}$$

To find the solutions, in the complex plane, of this exponential equation is not an easy problem, and we do not proceed this way. Instead we study a simplified, but practically important, case and then evaluate the perturbations around this special solution.

We assume $L = L*$, where $L*$ is the luminosity which gives a mean temperature of the Earth equal to $T_e = \overline{T}$, assuming a mean albedo of $A = 0.5$. For this value of the luminosity, and for these values of the albedos, it follows that

$$\rho_b(L*) = \rho_w(L*), \tag{15}$$

and thus that

$$y_0 = 0, \tag{16}$$

which means

$$a_w^0 = a_b^0. \tag{17}$$

Furthermore,

$$\eta_w(L*) = -\eta_b(L*), \qquad \eta_w(L*) > 0, \tag{18}$$

(see in De Gregorio et al., 1992, for the explicit formulas). System (13), therefore, simplifies to

$$\begin{aligned} \dot{x}(t) &= -\frac{a_b^0\gamma}{p - z_0}[x(t) + y(t)] - a_b^0(p - z_0)\eta_w[x(t-\tau) - y(t-\tau)], \\ \dot{y}(t) &= -\frac{a_w^0\gamma}{p - z_0}[x(t) + y(t)] - a_b^0(p - z_0)\eta_w[x(t-\tau) - y(t-\tau)]. \end{aligned} \tag{19}$$

What is important in this new system is not the fact that there are fewer independent parameters, but the fact that the matrix of coefficients is now symmetric and the characteristic equation factorizes. The characteristic equation of this system is

$$\left(\lambda + \frac{z_0\gamma}{p - z_0}\right)\left[\lambda + 2(p - z_0)\eta_w a_w^0 e^{-\lambda\tau}\right] = 0, \tag{20}$$

from which it follows that one eigenvalue is negative and the other is given by the exponential equation

$$\lambda + be^{-\lambda\tau} = 0, \qquad b = 2(p - z_0)\eta_w a_w^0. \tag{21}$$

This equation has infinite solutions in the complex plane. In particular, it has purely imaginary solutions for

$$\tau_k = \left(\frac{\pi}{2} + 2k\pi\right)\frac{1}{b}, \qquad k = 0, 1, 2, \ldots. \tag{22}$$

As we will see more clearly later, to these values correspond periodic solutions of system (9). Thus, $\tau_0$ is the value of the delay for which the eigenvalues cross the imaginary axis of the complex plane; at this value the stationary solution $(a_w^0, a_b^0)$ loses its local asymptotic stability, and a periodic solution appears around the stationary solution. It is in fact very easy to understand the behavior of the present system. If we consider the sum of the two equations in (19), we obtain for the variable $u = x + y$ the equation

$$\dot{u}(t) = -\frac{z_0\gamma}{p - z_0}u(t). \tag{23}$$

This is a non-delayed equation whose solutions go exponentially to zero, that is, $x(t)$ goes to $-y(t)$. Thus, the two equations in (19), in the limit, become

$$\begin{aligned} \dot{x}(t) &\to -2a_w^0(p - z_0)\eta_w x(t - \tau) = -bx(t - \tau), \\ \dot{y}(t) &\to -2a_w^0(p - z_0)\eta_w y(t - \tau) = -by(t - \tau), \end{aligned} \tag{24}$$

that is, identical delayed equations.

The factorization of the characteristic equation describes, for the factor

$$\lambda + \frac{z_0\gamma}{p - z_0}, \tag{25}$$

the behavior of equation (23), while the factor

$$\lambda + be^{-\lambda\tau} = \lambda + 2(p - z_0)\eta_w a_w^0 e^{-\lambda\tau} \tag{26}$$

describes equations (24). The equations in (24) are in fact the same as the linear approximation of a classical logistic equation with constant time lag (Cushing, 1977). For the logistic equation it has been proved that the stationary solution is globally attracting for $\tau < \frac{3}{2}(1/b)$, that all the solutions oscillate for $\tau > (1/e)(1/b)$. We obtain numerically similar results for our system (8). For the system of linear approximation (23–24) we can say that, while the variable $u(t)$ goes exponentially to zero, such that $a_b(t) + a_w(t) \to z_0$, the other two variables oscillate on the line $z = z_0$ and, for $\tau = \tau_0$, they behave periodically. The period of this periodic oscillation is $T = 4\tau_0$. For $\tau > \tau_0$ the solutions of the linear system diverge exponentially from the stationary solution, and the linear approximation cannot give any further useful information. We found that with our system, for $\tau > \tau_0$, there still is a periodic solution. The numerical results show also that the relation between the delay and the period is still approximately valid also for $\tau >> \tau_0$.

The present analysis is valid for $L = L*$, but we will see that, for $L$ around $L*$, nothing changes qualitatively. And, indeed, our numerical results also give periodic orbits for $L \neq L*$.

What happens is that, even though, for $\tau > \tau_0$, one eigenvalue of the characteristic equation has a positive real part and the solutions expand exponentially away from the stationary solution, the phase space available to the system is bounded, that is,

$$0 \leq a_b + a_w < p, \qquad a_b, a_w \geq 0, \tag{27}$$

so this exponential divergence in any case has to end and the amplitude of the oscillations has to remain bounded. The long-time behavior of such a system, constrained in a bounded region, lacking a stationary or periodic attracting solution, can be very complicated, for example of a chaotic type. In our case, generally, these attracting periodic solutions exist. Concerning the critical value of the delay $\tau_0$, as given below in (28), it results that it is of the order of $1/\gamma$. Now, if we recall that we already assumed $1/\gamma$ as a possible time scale of the system, we can say that these periodic solutions of the present model of feedback between the solar luminosity and the environmental biosphere can appear only if the delay in the growth-rate response to external variations of luminosity is of the order (or larger) than the response time of the system.

We now reconsider system (13) for $L \neq L*$. Introducing the parameter $x = L/L*$, and assuming that $\alpha$ is small (which indicates strong diffusion), the previous discussion is essentially unchanged. The important fact is that, with errors of the second order in $(x - 1)$, the characteristic equation of the system can again be factorized, and the critical value of the delay now has a factor $1/x$ in it. Thus, by increasing the solar luminosity, there is an easier possibility for the system to show oscillating solutions.

The numerical simulations are again in good agreement with these results. For completeness we present the explicit dependence of the critical value of $\tau_0$ as a function of the parameters of the system;

$$\tau_0 = \frac{\pi}{2} \times \frac{\{1 - [(T_0 + \overline{T})/4]^2 \delta\alpha^2 (1 - 2A_b)^2\}^2}{x^2\gamma\alpha(1-\alpha)\delta\,[(T_0 + \overline{T})/4]^2 (1 - 2A_b)\{1 - \delta\,[(T_0 + \overline{T})/4]^2 \alpha^2(1 - 2A_b)^2 - \gamma\}};$$

$$A_b + A_w = 1. \tag{28}$$

## 4. Integrated Delay

From the physical point of view it is certainly not realistic to consider a constant time lag. We would be more satisfied with a formulation of the problem in which the growth-rate response to external variations depends on the previous history of the system, with an influence of the past going exponentially to zero with time. The differential system (8) would thus become an integrodifferential system with $\beta(t-\tau)$ substituted by an integral of Volterra type, that is,

$$\int_{-\infty}^{t} k(t,s)\beta(s)\,ds. \tag{29}$$

Our integrodifferential system is

$$\dot{a}_i(t) = a_i(t)\left\{[p - z(t)]\int_{-\infty}^{t} k(t,s)\beta_i(s)ds - \gamma\right\}, \qquad i = b, w. \tag{30}$$

Without considering the theory in general, we chose two particular, but very illuminating, forms for the kernel $k$. We suppose first that $k(t,s) = k(t-s)$, and then we define $k(t)$, for $t \geq 0$, as

$$k(t) = \frac{1}{\tau}e^{-t/\tau} \tag{31}$$

or

$$k(t) = \frac{1}{\tau^2}te^{-t/\tau}. \tag{32}$$

The two kernels are normalized so that

$$\int_{0}^{\infty} k(t)\,dt = 1. \tag{33}$$

The first kernel (31) corresponds to a maximum contribution to the variation of the growth rate from the current value of the species fractional coverage, with the contribution decaying exponentially to zero for the past values. For this reason this delay is called *weak* (see Cushing, 1977). The kernel (32), in contrast, has a maximum for $t = \tau$, then decays exponentially to zero for $t \rightarrow \infty$, and is zero for $t = 0$. This is a smooth representation of the constant time lag, and for this reason this delay is called *strong*. We would expect that the weak delay would give a behavior more connected to the nondelayed equation, while the strong delay would give a behavior gentler than for the constant time lag. Our results show that the stationary solution does not lose its local asymptotic stability for a weak delay, while the critical value of the parameter $\tau_0$ for the strong delay is slightly larger than that for the constant time lag, so that the stationary solution loses its local asymptotic stability for a larger delay. To be precise, the characteristic equation for system (30) and a weak delay is

$$\lambda^2(\lambda\tau + 1) + \lambda\left[(\lambda\tau + 1)\frac{z_0\gamma}{p - z_0} + (p - z_0)\left(2\nu y_0^2 + \eta_w a_w^0 - \eta_b a_b^0\right)\right] + 2\gamma a_b^0 a_w^0(\eta_w - \eta_b) = 0. \tag{34}$$

The study of the Rooth-Hurwitz conditions for this equation would give the necessary and sufficient conditions for the stability of the stationary solutions. In fact, we can avoid this analysis by taking $L = L*$. Also in this case we have the factorization of the characteristic equation

$$\left(\lambda + \frac{z_0\gamma}{p - z_0}\right)\left[\lambda(\lambda\tau + 1) + 2(p - z_0)\eta_w a_w^0\right] = 0, \tag{35}$$

and we can see that the eigenvalues always have a negative real part, and so, for any value of the delay $\tau$, the stationary solution $(a_w^0, a_b^0)$ retains its local asymptotic stability. The numerical simulations of the nonapproximated system confirm the conclusion; even for a very high delay the system always goes quickly to the stationary solution. For $L \neq L*$, and $L \approx L*$, in the case of weak delay, there is nothing new qualitatively.

The characteristic equation for system (30), in the case of strong delay and $L = L*$, is

$$\lambda^2(\lambda\tau + 1)^2 + \lambda\left[(\lambda\tau + 1)^2\frac{z_0\gamma}{p - z_0} + (p - z_0)\left(2\nu y_0^2 + \eta_w a_w^0 - \eta_b a_b^0\right)\right] + 2\gamma a_b^0 a_w^0(\eta_w - \eta_b) = 0. \tag{36}$$

Also in this case we can avoid the more difficult study of the Rooth-Hurwitz conditions by assuming $L = L*$. The factorization of the characteristic equation is

$$\left(\lambda + \frac{z_0\gamma}{p - z_0}\right)\left[\lambda(\lambda\tau + 1)^2 + 2(p - z_0)\eta_w a_w^0\right] = 0. \tag{37}$$

Here we have to study the conditions for a cubic and not a quartic equation, while the other eigenvalue is always negative. The cubic equation has a purely imaginary solution for $\tau = \tau_0$;

$$\tau_0 = \frac{1}{(p - z_0)\eta_w a_w^0}. \tag{38}$$

This eigenvalue crosses the imaginary axis with a velocity different from zero. Thus, knowing that, for periodic solutions, the equation

$$\dot{x}(t) = -\alpha x(t)\int_{-\infty}^{t} k(t - s)f(x(s))\,ds, \qquad k(t) = \frac{t}{\tau^2}e^{-t/\tau}, \tag{39}$$

is equivalent to a system of ODEs, and we can conclude that the delay $\tau = \tau_0$ corresponds to a Hopf bifurcation. When the stationary solution $(a_w^0, a_b^0)$ loses its local attractivity, a periodic solution appears that is locally attracting and, at the bifurcation point, has a period of $T = 2\pi\tau_0$. Also in this case, for $L \neq L*$, but in a range of values around $L*$, and for $\alpha \ll 1$, we have that the critical value is given by

$$\tau_0 = \frac{2}{x^2\gamma(1 - \gamma)\delta\left[(T_0 + \overline{T})/4\right]^2\alpha(1 - \alpha)(A_w - A_b)}. \tag{40}$$

## 5. Numerical Results

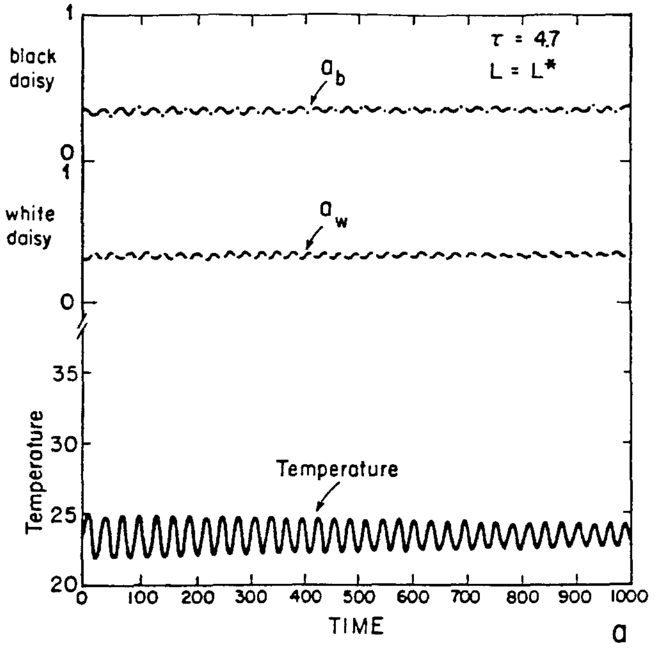
The numerical simulations giving the evolution of our delayed systems, without linear approximations, are in good agreement with the analytical results of the previous sections. If different values are not explicitly indicated, the luminosity is $L = L*$, the mortality coefficient $\gamma = 0.2$, the fertility $p = 1$, the natality coefficient $\beta = \beta_0$ ($\beta_0$ is the value in the text after formula (6) giving $\beta_0(\overline{T}) = 1$), the coefficient of diffusion $\alpha = 0.3$, and the albedos are $A_b = 0.25$ and $A_w = 0.75$. Figure 1 shows the existence of periodic solutions, in the case of the strong delay, with the value of the delay in good agreement with that given by formula (40). Also, the period of the solution is the predicted one, and furthermore, as expected, an increase of solar luminosity makes the transition to periodic solutions easier (compare Fig. 1b with Fig. 2). An increase of the delay makes the period longer and enhances the amplitude of the oscillations (Figs. 3–4). A phenomenon that would be interesting to analyze in more detail is that of the doubling of orbits shown in Fig. 4b. A further increase (Fig. 5) of the delay makes the period still longer and the amplitude of the oscillations larger. It is easy to understand that if all the ground is fertile ($p = 1$), then the competition between the two species produces rapid transitions from nearly all the ground covered by white daisies to the opposite case of nearly all ground covered by black daisies (the exact fractional area depending on the coefficient of death $\gamma$). Thus, if we reduce by a factor of 2, 4, . . . the size of fertile ground ($p = 0.5, 0.25$) and, in correspondence, increase $\beta$ by a factor of 2, 4, . . . (this is necessary in order to have $p - \gamma/\beta > 0$), we have a more realistic situation. These results are given in Figs. 6–9 for different values of the parameters. Note that the phase space is now reduced, and so there is a rescaling of the axes. Observe that the critical delay has to be rescaled. Figs. 6–9 also show the predicted oscillations of $(a_w(t), a_b(t))$ about the line

$$a_w + a_b \sim z_0.$$

We now comment on the results for the constant time lag case. The general remark to restate is that the critical delay is now shorter than the strong delay case. And, in fact, Fig. 10b, with a delay $\tau = 20$, is similar to Fig. 7, which corresponds to a delay of $\tau = 30$. But if we allow $p$ to be equal to 1, we not only have very large oscillations (Fig. 11), but, even after long integration time, there still remains a nonperiodic behavior that should be further analyzed to see if there is a chaotic behavior. In any case, if we either decrease the difference of the albedos between the daisies or decrease the value of $p$, we still obtain the expected periodic solutions, as Figs. 12–15 show. Furthermore, the particular behavior illustrated in Fig. 11 disappears if we consider a luminosity different from the reference luminosity $L*$ (see Fig. 15). Observe that for $L = 1.1L*$ there is an extinction of the black daisies.

## 6. Conclusions

The introduction of delay into the differential system that describes "Daisyworld" gives a richer structure to the system. Without delay we saw in De Gregorio et al.



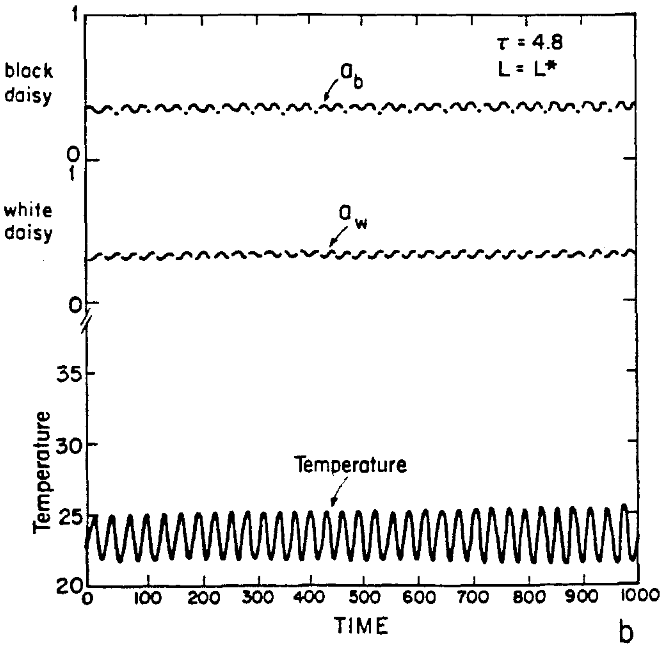




**Fig. 1.** Earth temperature perturbation and population fractional coverage $a_w$, $a_b$ as functions of time, for the strong delay model; $\tau = 4.7$ and $\tau = 4.8$, respectively.

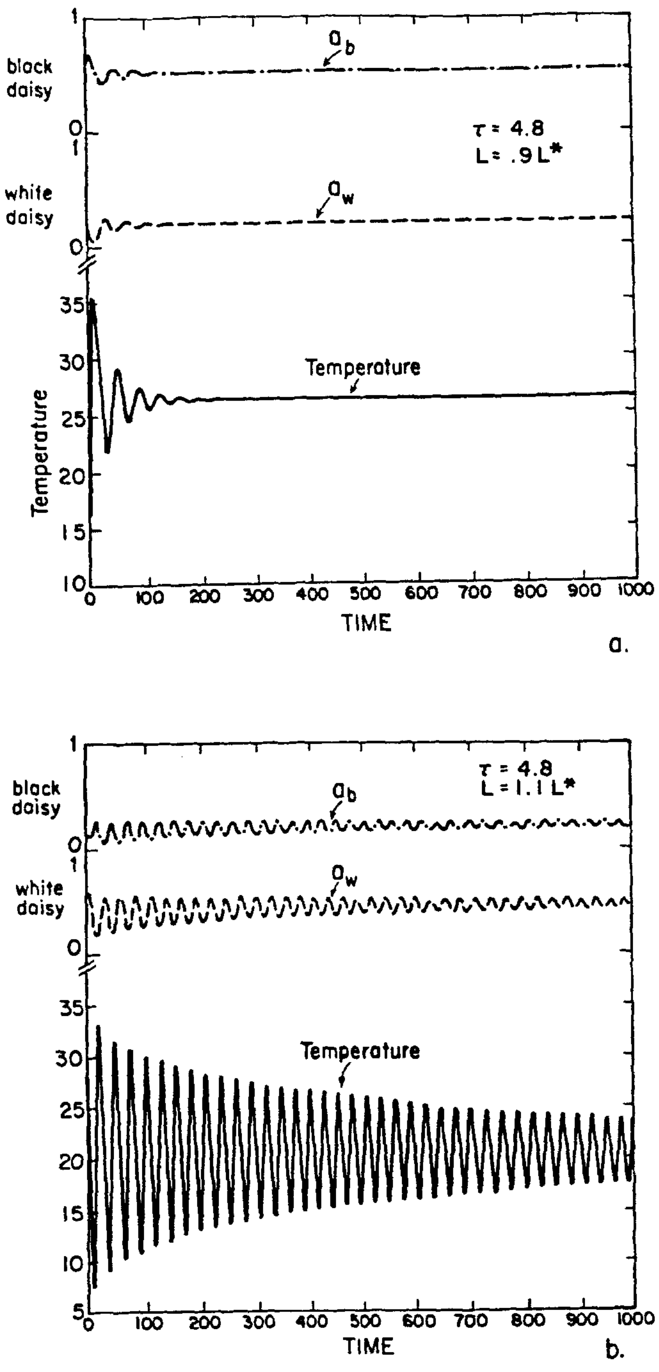



**Fig. 2.** Earth temperature perturbation and population fractional coverage $a_w$, $a_b$ for the strong delay model; $\tau = 4.8$, with luminosity $L = .9L*$ and $L = 1.1L*$ respectively.

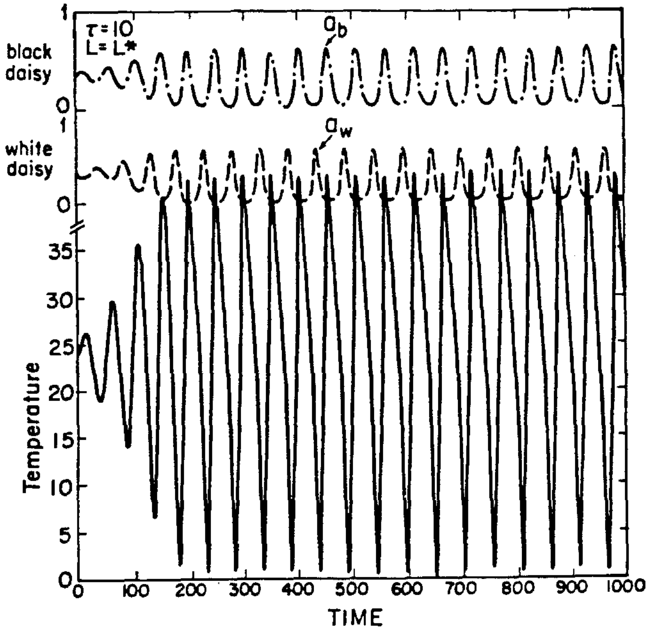



**Fig. 3.** Amplification of the perturbation in Fig. 1 when the delay is increased to $\tau = 10$. Observe the corresponding doubling of the period of oscillations.

(1992) that, for a suitable range of values for the parameters, the feedback between the solar luminosity and the biosphere is very efficient, and the temperature of the Earth is practically constant for sustained variations of solar luminosity. With the inclusion of the delay we can have, for the same values of the parameters, oscillations of the temperature of the Earth, reaching very critical values. But if we consider that not all the surface of the Earth can be covered by the biosphere, that the oceans have a behavior different from the ground, and that the Earth is curved, the oscillations, even if still present, *might* be more contained. [Some of these important factors (presence of oceans, the curvature of the Earth) need to be explored.]

In the present work, as a more physical representation of the real world, we considered a value for the fertility of the Earth different from one, in fact much less than one, and we then rescaled the natality coefficient. As a consequence, if the delay is larger than a critical value, both models, the constant time lag model and the strong delay model, give very similar periodic behavior. In the weak delay model, however, also for very large delay we did not obtain periodic orbits, but only vanishing oscillations.

An aspect that can be an object of future work in the delayed models is also the possibility of chaotic behavior over a certain range of values for the parameters. For the moment, there is an indication that this can happen only in the constant time lag case and for large values of the parameters. In any case, many people expect this chaotic behavior as a possible feature of the Earth's climate.

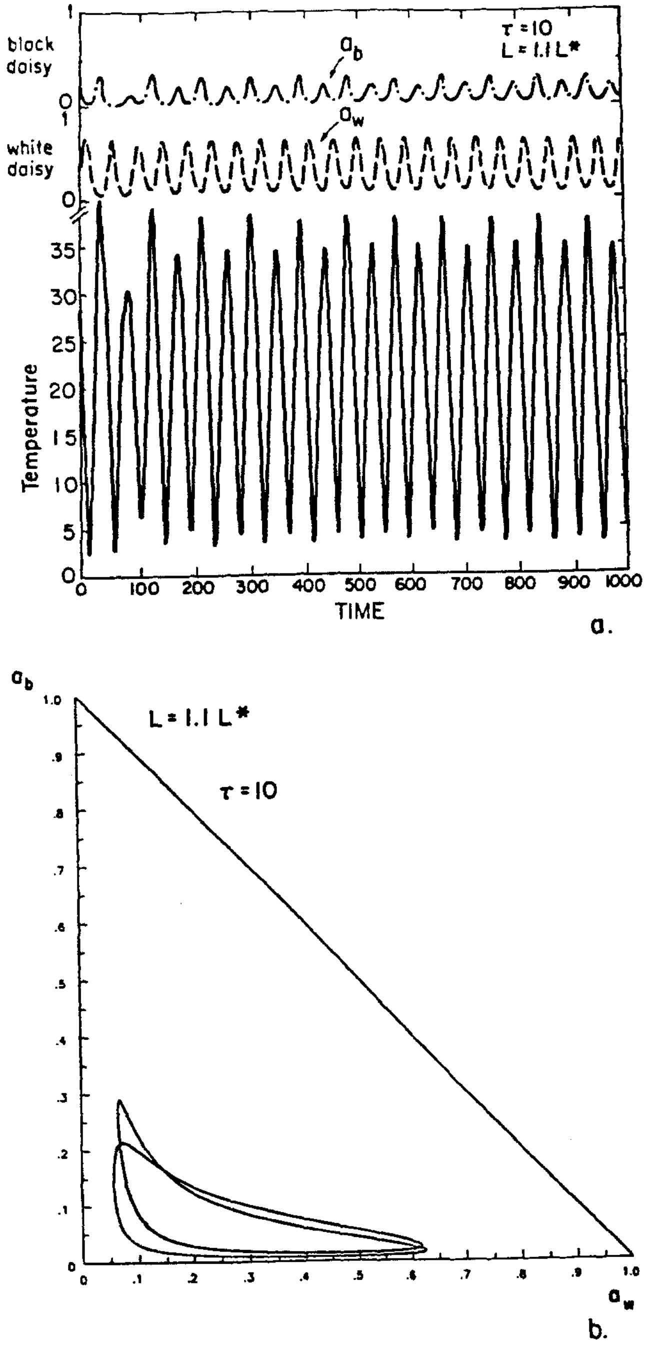



**Fig. 4a, b.** Perturbations and phase space plot $(a_w, a_b)$ for $\tau$ = 10 and $L = 1.1L*$.

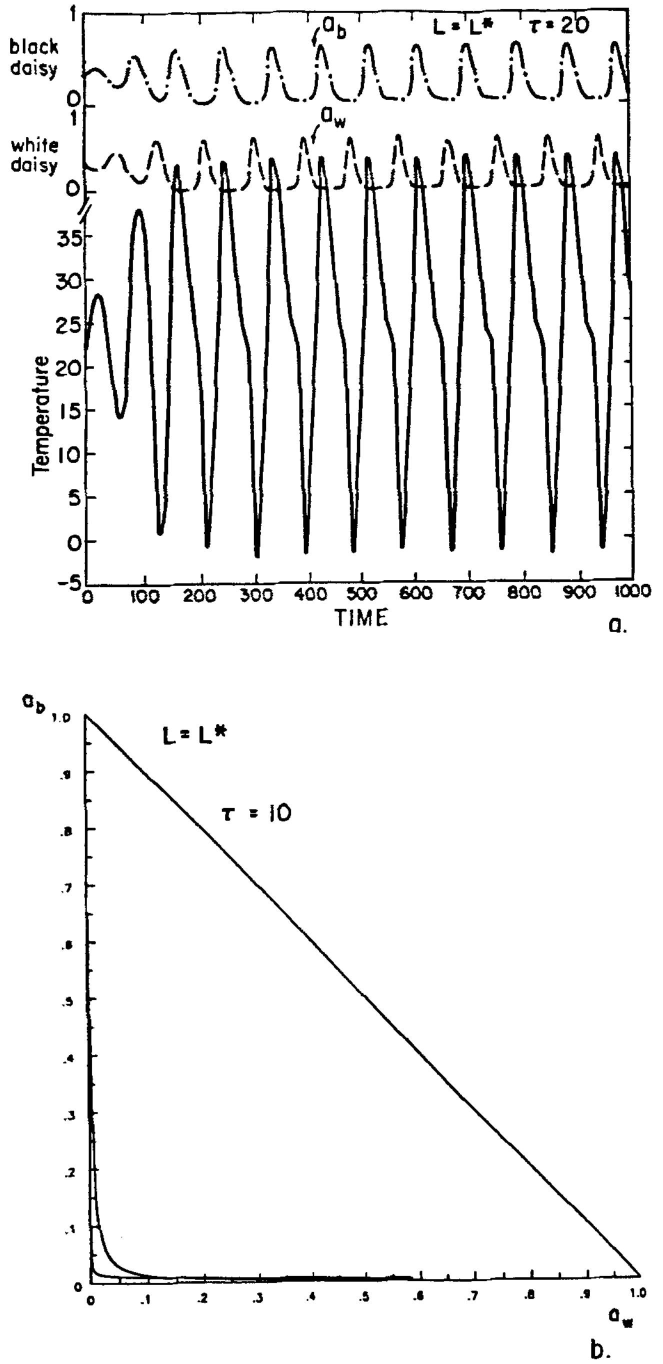
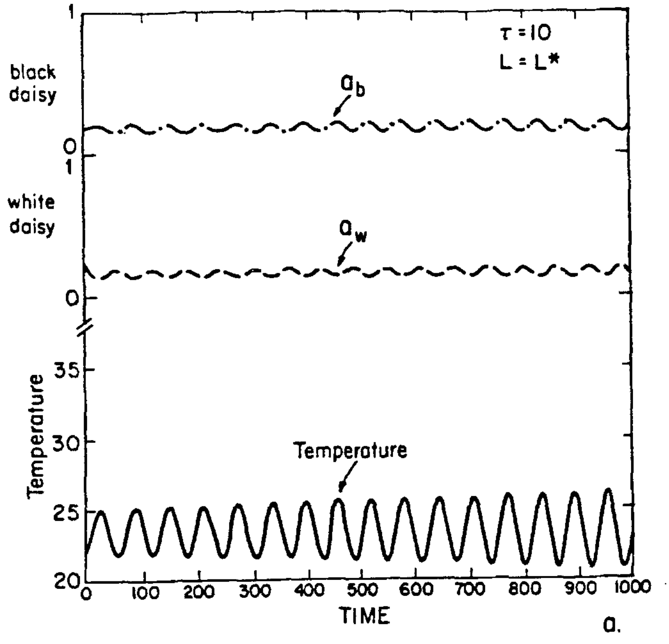



**Fig. 5a, b.** Perturbations and phase space plot ($a_w$, $a_b$) for $\tau = 20$ and $L = L*$.



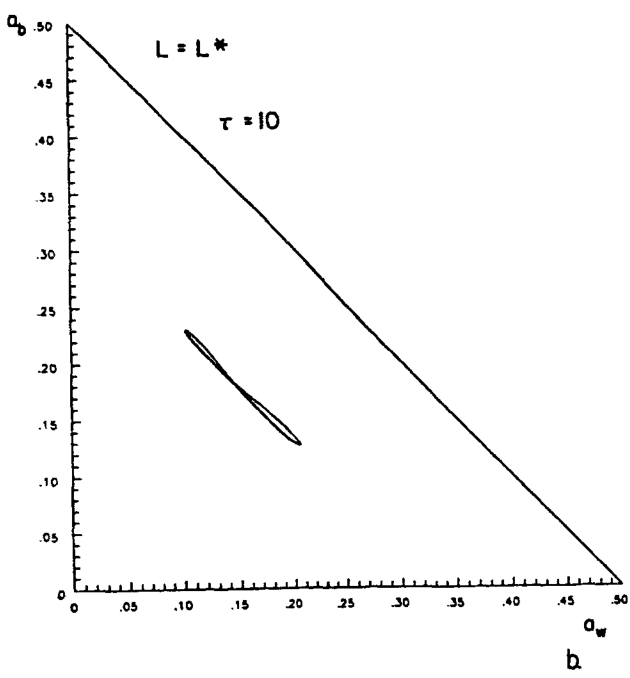




**Fig. 6a, b.** Perturbations and phase space plot ($a_w$, $a_b$) for $p = 0.5$, $\beta = 2\beta_0$, and $\tau = 10$; note the rescaling of the axis $a_w$, $a_b$.

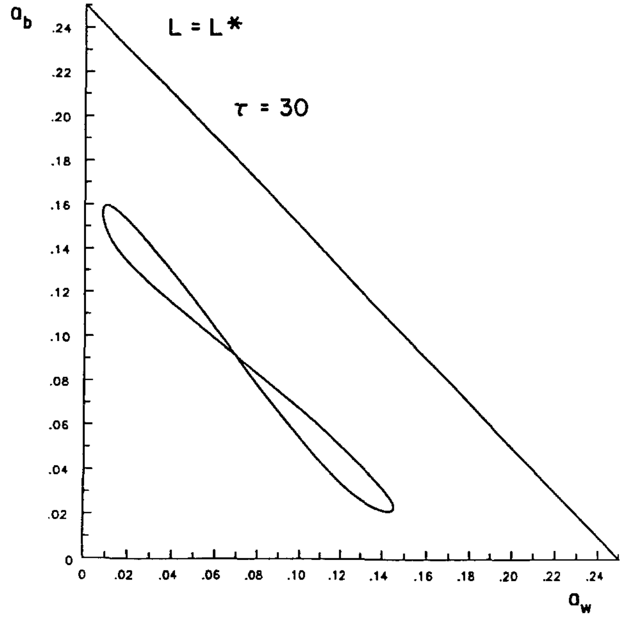



**Fig. 7.** Phase space plot ($a_w$, $a_b$) for a further rescaling: $p = 0.25$, $\beta = 4\beta_0$, $\tau = 30$.

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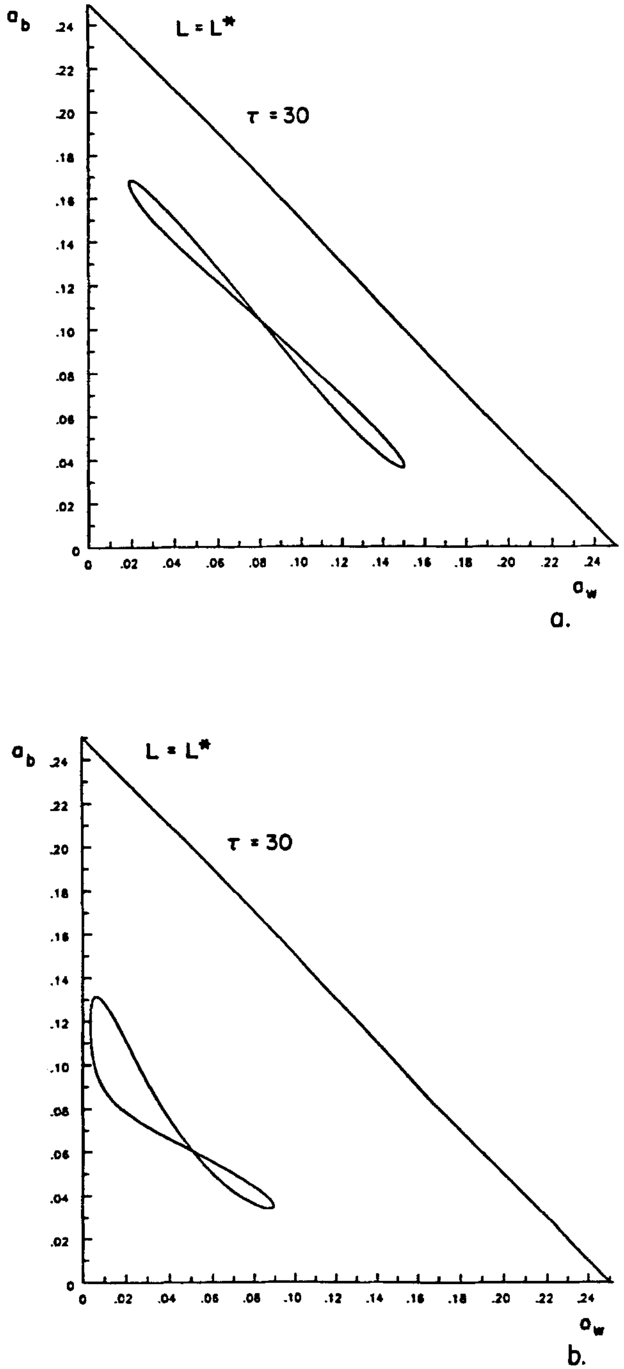



**Fig. 8.** Modification of the phase space plot of Fig. 7 when $\gamma$ is lowered from 0.2 to 0.15 (Fig. 8a) and when in addition $\alpha$ is increased from 0.3 to 0.4 (Fig. 8b).

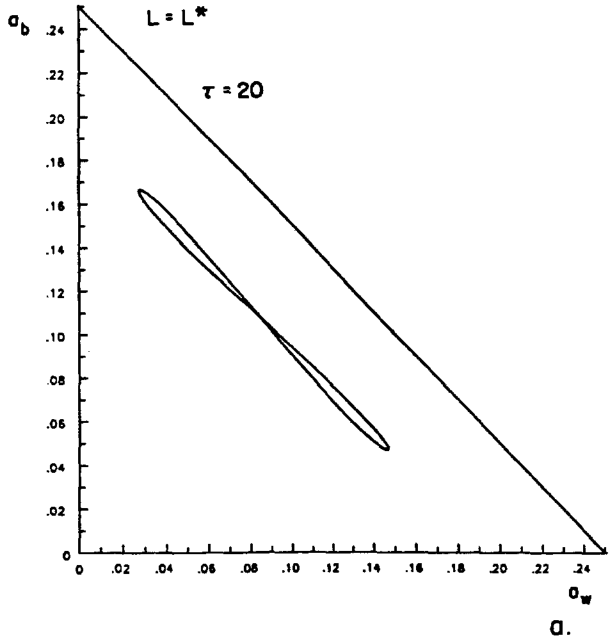



**Fig. 9a.** Perturbations for $p = .25$, $\beta = 6\beta_0$, $\tau = 20$, and $\alpha = 0.3$.

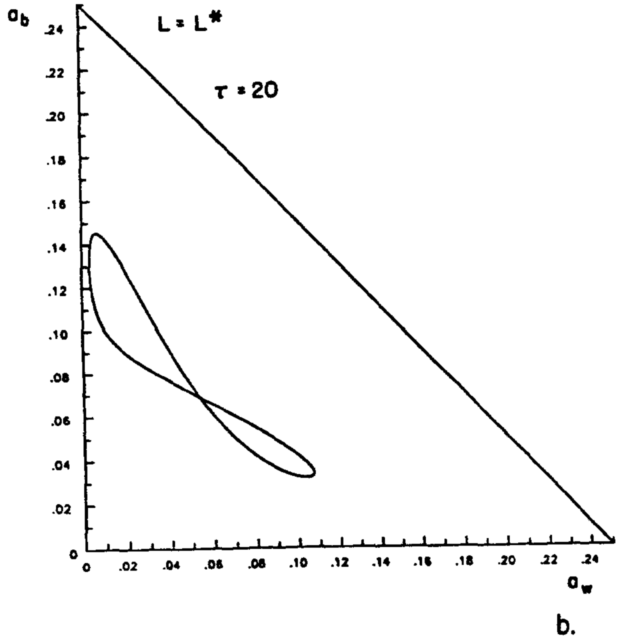



**Fig. 9b.** Phase space plot ($a_w$, $a_b$) with the parameters as in Fig. 9a, but $\alpha = 0.4$.

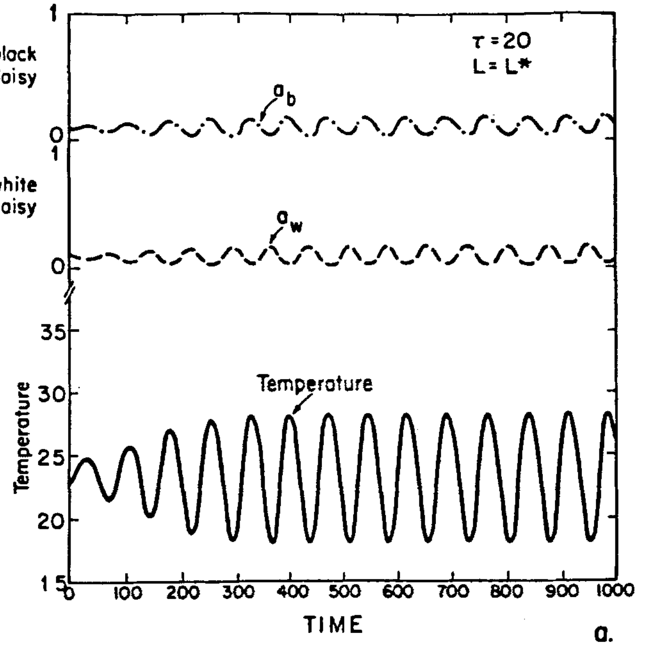



**Fig. 10a.** Temperature perturbation and population fractional coverage $a_w$, $a_b$ as functions of time, for the constant time lag model and $\tau = 20$, $p = 0.25$, and $\beta = 4\beta_0$.

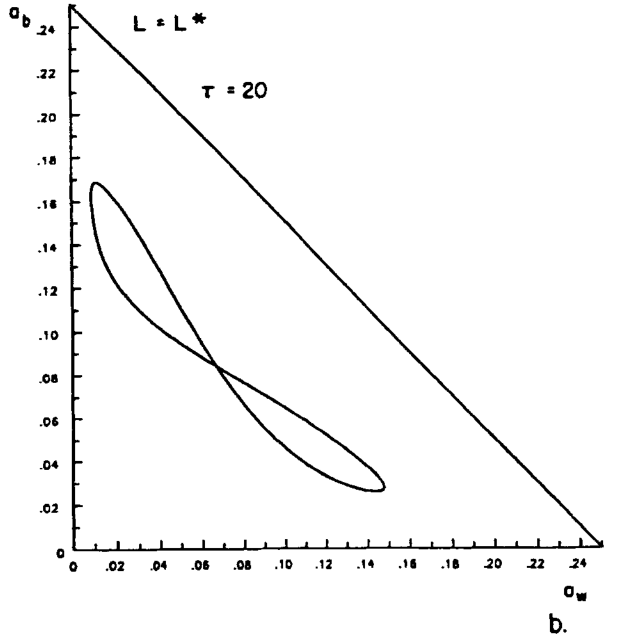
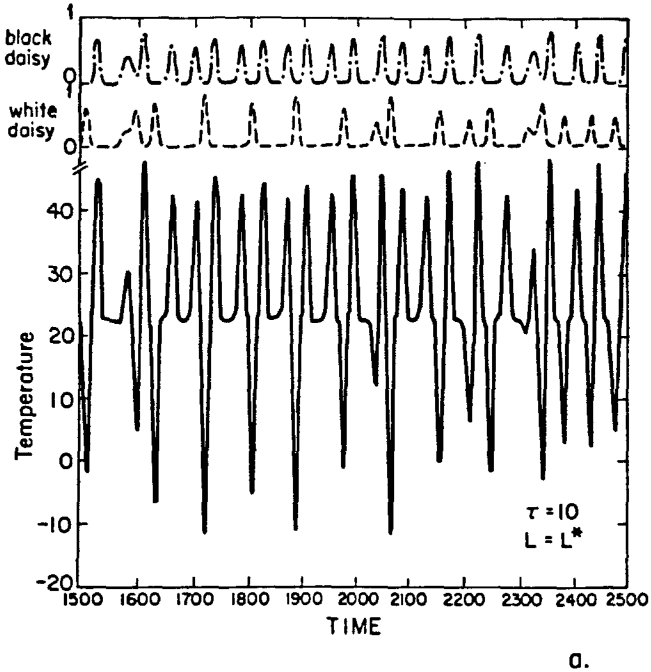



**Fig. 10b.** Phase space plot ($a_w$, $a_b$) corresponding to Fig. 10a.



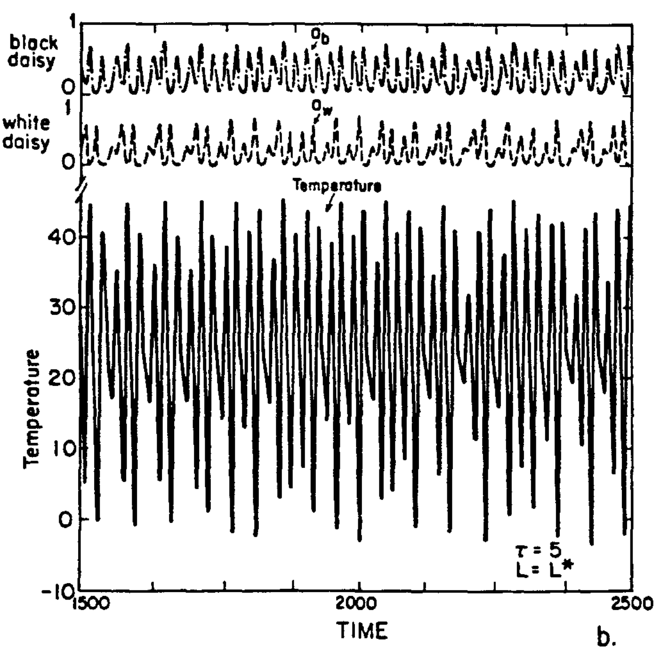
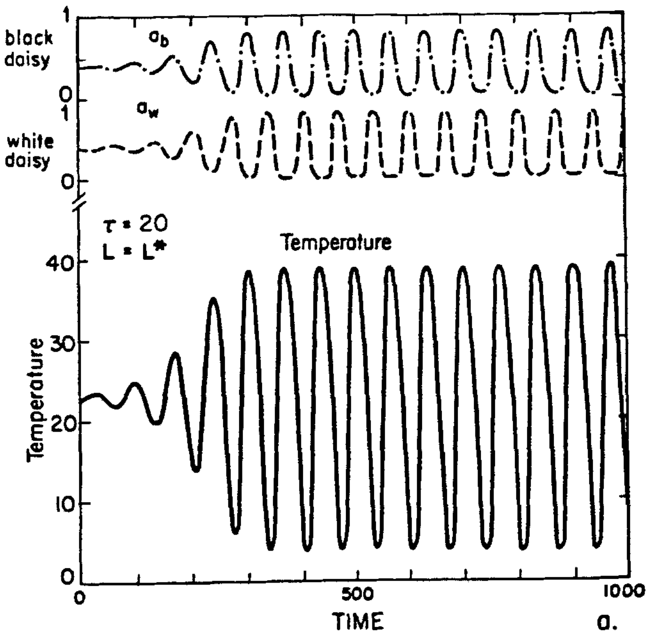




**Fig. 11a, b.** The shift to an apparently chaotic behavior when the fertility is raised to $p = 1$ with $\beta = \beta_0$ and $\tau = 10$ and $\tau = 5$, respectively.



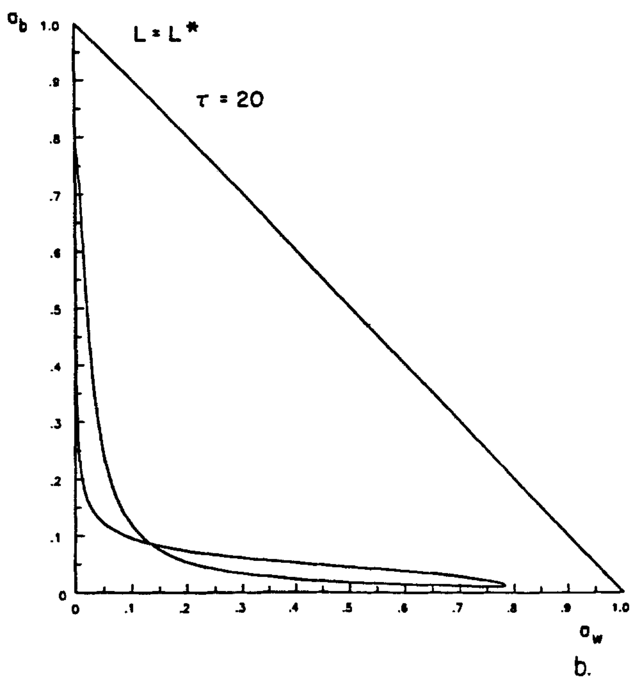




**Fig. 12a, b.** Perturbations and phase plot $(a_w, a_b)$ when the difference of the albedos is lowered from $A_b = 0.25$, $A_w = 0.75$ to $A_b = 0.35$, $A_w = 0.65$ with $\tau = 20$.

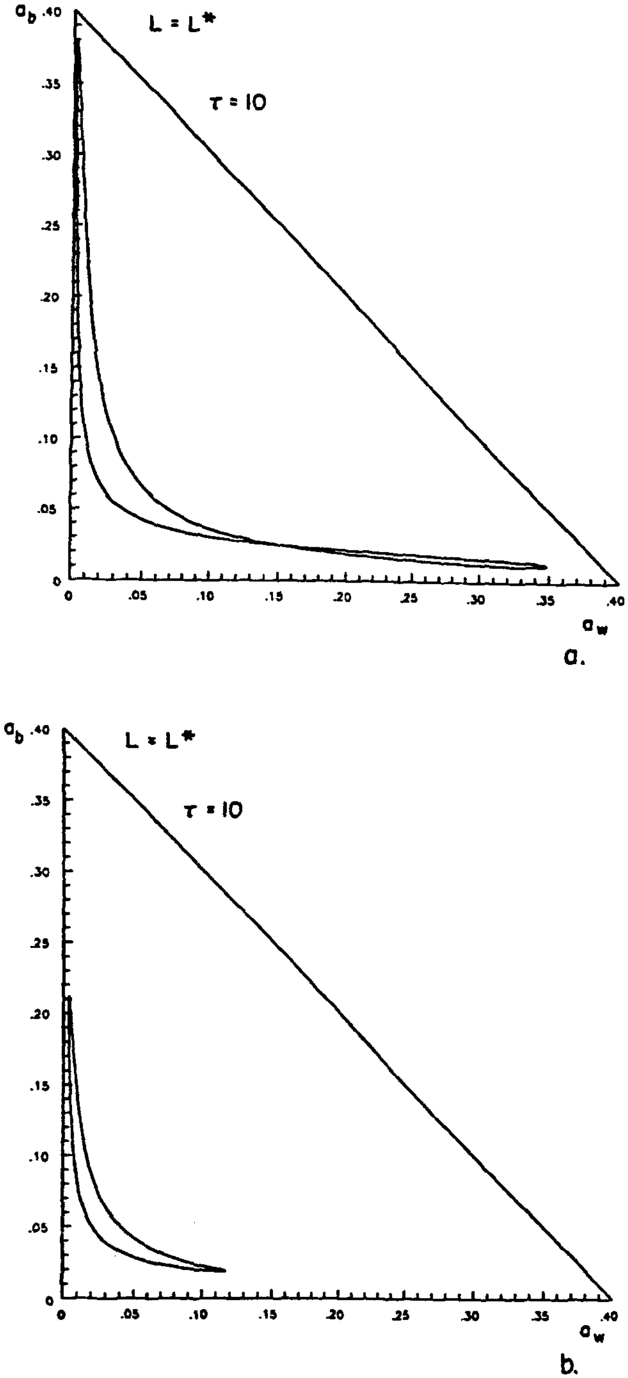



**Fig. 13a, b.** Phase space plots with $p = 0.5$, $\beta = 2\beta_0$, $\tau = 10$ and $\alpha = 0.3$ and $\alpha = 0.4$, respectively.

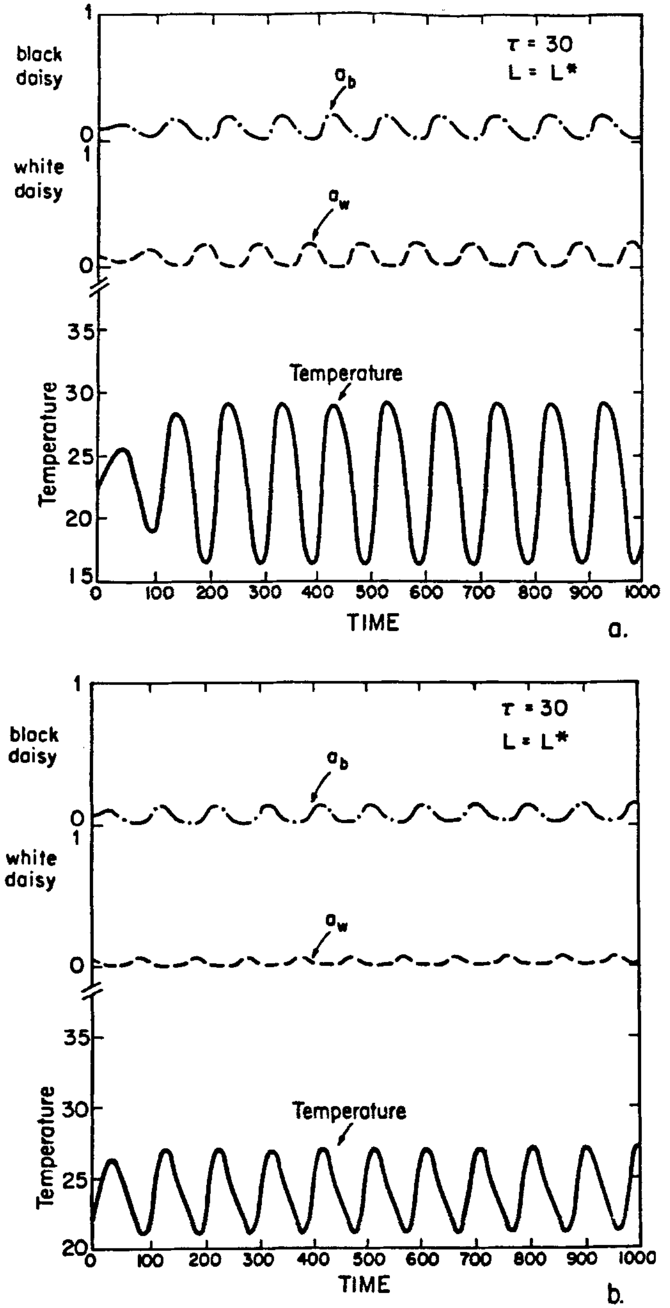



**Fig. 14a, b.** Slight modification of perturbations with respect to Fig. 10a, with $p = 0.25$, $\beta = 4\beta_0$, $\tau = 30$ and $\alpha = 0.3$ and 0.4 respectively.

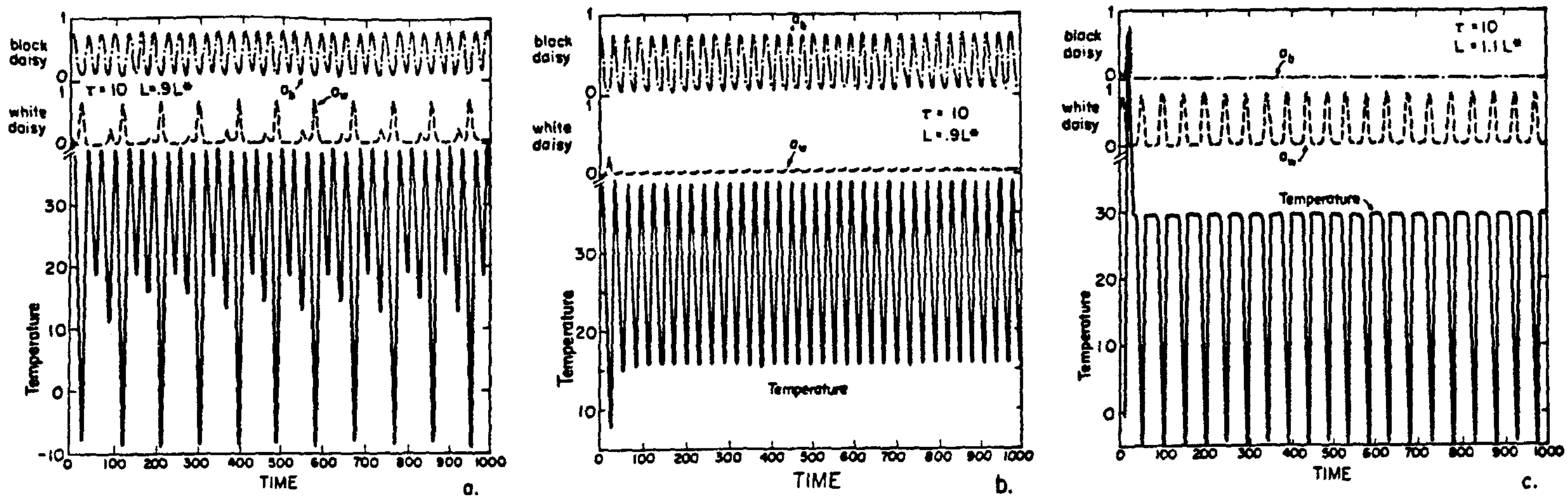



**Fig. 15a, b, c.** The shift of the apparently chaotic behavior of Fig. 11 to periodic behavior when $L$ is different from $L*$ and $p = 1$, $\beta = \beta_0$, and $\tau = 10$. In Fig. 15a, b, $L = 0.9L*$ and $\alpha = 0.3$, and 0.4, respectively. In Fig. 15c $L = 1.1L*$ and $\alpha = 0.3$.

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