Derivation of Slope Flow Equations Using Two Different Coordinate Representations

R. A. PIELKE AND M. SEGAL

Department of Atmospheric Science, Colorado State University, Fort Collins, CO 80523

R. T. MCNIDER

K. E. Johnson Environmental and Energy Center, University of Alabama, Huntsville, AL 35899

Y. MAHRER[1]

Cooperative Institute for Research in the Atmosphere, Colorado State University, Fort Collins, CO 80523

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ABSTRACT

This paper examines two coordinate representations for slope flow models, one a rotation of the coordinate axes, the other a generalized vertical coordinate transformation. An analytic solution is developed in both representations for a uniform slope to examine the differences due to slightly different forms of a generalized hydrostatic equation. For the first transformation, velocity accelerations in the direction of the generalized vertical coordinate are ignored, while for the second transformation, velocity accelerations perpendicular to the terrain are neglected. Surprisingly, only the period of flow oscillation and not the mean strength of the slope flow was changed in using the first coordinate representation instead of the second. Only for slopes greater than 45° does the difference in periods between the two transformations exceed 30%. Differences which may occur for nonuniform slopes, however, still need to be examined.

1. Introduction

As shown by Dutton (1976) and applied by Pielke and Martin (1981; 1983), the derivation of physically consistent mathematical relations for the equations of motion in different coordinate systems is facilitated by the use of tensor transformation procedures. In this paper we utilize this approach in order to derive equations in two distinct coordinate systems which can be used to simulate slope flow[2] in a two-dimensional model. The first transformation uses a generalized vertical coordinate, while the second applies a rotation of the Cartesian coordinate axes. Two generalized hydrostatic-like equations are defined for the two transformations. For the first transformation, velocity accelerations in the direction of the generalized vertical coordinate are ignored, while for the second, velocity accelerations perpendicular to the terrain are neglected. An outline of these transformations is given in Section 2; the derivation of the pertinent equations for two-dimensional drainage flow along a constant slope using these transformed systems along with the two different generalized hydrostatic equations are performed in Section 3. Analytic comparisons of the flow characteristics as represented in both transformations follow in Section 4. Conclusions of this paper are given in Section 5.

[1] Permanent affiliation: Seagram Centre for Soil and Water Science, Faculty of Agriculture, The Hebrew University of Jerusalem, Rehovot 76100, Israel.

[2] Following Fitzjarrald (1984), we refer to "slope flow" to indicate flow due to cooling along a slope as opposed to "drainage flow" which is a result of an initial buoyancy deficit.

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2. Methodology

As shown by Dutton (1976, p. 250), the equation of motion can be written in general form as

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^k \tilde{u}^i_{;k} = -\tilde{G}^{ik}\left(\frac{1}{\rho}\frac{\partial p}{\partial \tilde{x}^k} + \frac{\partial \Phi}{\partial \tilde{x}^k}\right) - 2\tilde{\epsilon}^{ijk}\tilde{\Omega}_j \tilde{u}_k + \tilde{f}^i, \quad (1)$$

where the second term on the left is the covariant derivative, $\tilde{G}^{ik}$ the metric tensor, $\tilde{\epsilon}^{ijk}$ the permutation tensor, $\Phi = gz$ the geopotential and $\tilde{f}^i$ the subgrid scale friction. The velocity components $\tilde{u}^i$ are in their contravariant form. The remaining variables have their usual definition.

In order to obtain the equations to be used for a slope flow representation, the following assumptions are made:

i) the Coriolis force can be neglected;
ii) the coordinate transformation is from the Cartesian system.

With these conditions, (1) can be rewritten as

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^k \tilde{u}^i_{;k} = -\frac{\tilde{G}^{ik}}{\rho}\frac{\partial p}{\partial \tilde{x}^k} - g\frac{\partial \tilde{x}^i}{\partial z} + \tilde{f}^i. \quad (2)$$

If the Boussinesq assumption is made, Eq. (2) becomes

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^k \tilde{u}^i_{;k} = -\frac{\tilde{G}^{ik}}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^k} - g\frac{\rho'}{\rho_0}\frac{\partial \tilde{x}^i}{\partial z} + \tilde{f}^i, \quad (3)$$

where $p' = p - p_0$ and $\rho' = \rho_0 - \rho$ represent deviations of pressure and density from a large-scale value. For shallow atmospheric systems $-g\rho'/\rho_0 \approx g\theta'/\theta_0$ (e.g., see Dutton and Fichtl, 1969; Pielke, 1984), so that (1) becomes

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^k \tilde{u}^i_{;k} = -\frac{\tilde{G}^{ik}}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^k} + g\frac{\theta'}{\theta_0}\frac{\partial \tilde{x}^i}{\partial z} + \tilde{f}^i. \quad (4)$$

Equation (4) can be used to obtain the form of the equation of motion in a transformation from the Cartesian system to any other representation in which a functional relation exists between the independent variables in the two systems.

In this paper, two transformations are examined which are, for simplicity, illustrated for two dimensional formulations. The extension to three dimensions is straightforward. The formulations are:

I.	II.
$\tilde{x}^1 = x$	$\tilde{x}^1 = x\cos\gamma + z\sin\gamma$
$\tilde{x}^3 = z - z_G(x)$	$\tilde{x}^3 = z\cos\gamma - x\sin\gamma$
$x = \tilde{x}^1$	$x = \tilde{x}^1\cos\gamma - \tilde{x}^3\sin\gamma$
$z = \tilde{x}^3 + z_G(\tilde{x}^1)$	$z = \tilde{x}^1\sin\gamma + \tilde{x}^3\cos\gamma$

where z_G is terrain height and $\gamma = \tan^{-1}[\partial z_G/\partial x]$ is the slope angle of the terrain (see Fig. 1). The independent variables x and z represent an unrotated Cartesian coordinate system.

Transformation I represents a nonorthogonal generalized vertical coordinate transformation similar to that discussed in detail in Pielke and Martin (1981; 1983) where the $\tilde{x}^3$ coordinate is parallel to the gravity vector and $\tilde{x}^1$ is along the terrain slope.

Transformation II represents an orthogonal rotation in which $\tilde{x}^1$ is parallel and $\tilde{x}^3$ perpendicular to the terrain. Both transformations are illustrated in Fig. 2.

The orthogonal rotation is of the form commonly used to develop idealized analytic models of slope flow as summarized, for example, by Mahrt (1982) and applied by McNider (1982). In Mahrt's paper he suggests that one advantage of such a coordinate transformation is that the

> gravitational force perpendicular to the ground is approximately balanced by the pressure gradient force, while the component of the gravitational force parallel to the slope is not balanced and leads to downslope acceleration.

This type of separation into a hydrostatic part and a nonhydrostatic component is of substantial usefulness in developing analytic (and numerical) slope flow models and its application and generalization will be explored in this paper using Transformations I and II.

3. Transformations

a. Transformation I

Using the definition of $\tilde{G}^{ik}$, $\partial\tilde{x}^i/\partial z$ and covariant differentiation (see Pielke, 1984), and Transformation I, we obtain

$$\tilde{G}^{ik} = \begin{bmatrix} 1 & -\dfrac{\partial z_G}{\partial x} \\ -\dfrac{\partial z_G}{\partial x} & \left(\dfrac{\partial z_G}{\partial x}\right)^2 + 1 \end{bmatrix}; \quad \frac{\partial \tilde{x}^i}{\partial z} = (0, 1).$$

Eq. (4) can be written in component form for a constant slope angle as:

$$\frac{\partial \tilde{u}^1}{\partial t} + \tilde{u}^j\frac{\partial \tilde{u}^1}{\partial \tilde{x}^j} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^1} + \frac{1}{\rho_0}\frac{\partial z_G}{\partial x}\frac{\partial p'}{\partial \tilde{x}^3} + \tilde{f}^1, \quad (5)$$

$$\frac{\partial \tilde{u}^3}{\partial t} + \tilde{u}^j\frac{\partial \tilde{u}^3}{\partial \tilde{x}^j} = +\frac{1}{\rho_0}\left\{\frac{\partial z_G}{\partial x}\frac{\partial p'}{\partial \tilde{x}^1} - \left[\left(\frac{\partial z_G}{\partial x}\right)^2 + 1\right]\frac{\partial p'}{\partial \tilde{x}^3}\right\} + \frac{g\theta'}{\theta_0} + \tilde{f}^3. \quad (6)$$

As emphasized by Pielke and Martin (1983), this type of coordinate transformation has considerable utility because a type of hydrostatic assumption can

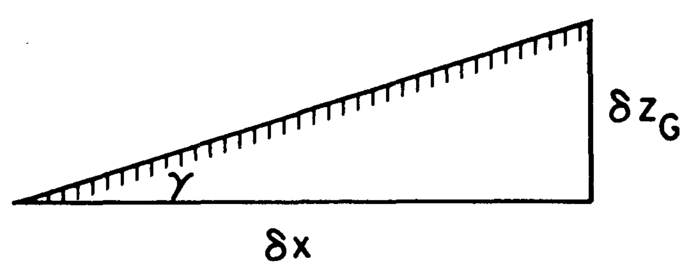

FIG. 1. The slope of terrain in terms of change in elevation δz_G and horizontal distance δx.

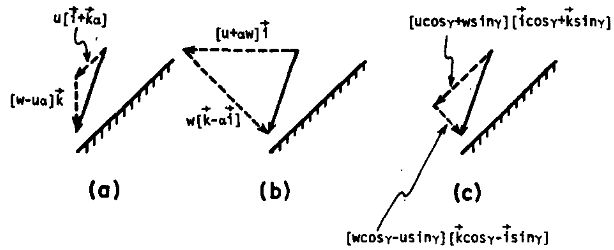

FIG. 2. Representations of a drainage flow wind using the contravariant (a) and covariant (b) forms of the velocity components derived from Transformation I, and (c) using the velocity components obtained by the orthogonal rotation Transformation II. The magnitude and direction of the components of the vector in the different representations have been given in terms of the Cartesian velocity components u and w, and basis vectors **i** and **k**. The slope $\partial z_G/\partial x = \alpha$ and γ is the slope angle.

be assumed valid in the $\tilde{x}^3$ direction but accelerations are still explicitly resolved in the terrain-parallel direction, which, for nonzero slope, has a component in the vertical direction. This form of hydrostatic representation is different from that suggested by Mahrt (1982), referenced earlier in this paper.

If the assumption is made in the $\tilde{x}^3$ direction that $d\tilde{u}^3/dt$ and $\tilde{f}^3$ are small relative to the other terms (i.e., a sort of generalized hydrostatic assumption), (5) and (6) reduce to

$$\frac{\partial \tilde{u}^1}{\partial t} + \tilde{u}^1 \frac{\partial \tilde{u}^1}{\partial \tilde{x}^1} + \tilde{u}^3 \frac{\partial \tilde{u}^1}{\partial \tilde{x}^3} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^1} + \frac{1}{\rho_0}\frac{\partial z_G}{\partial x}\frac{\partial p'}{\partial \tilde{x}^3} + \tilde{f}^1, \quad (7)$$

$$\frac{\partial p'}{\partial \tilde{x}^3} = \frac{1}{\left[\left(\frac{\partial z_G}{\partial x}\right)^2 + 1\right]}\left(\frac{g\theta'}{\theta_0}\rho_0 + \frac{\partial z_G}{\partial x}\frac{\partial p'}{\partial \tilde{x}^1}\right). \quad (8)$$

Inserting (8) into (5) and rearranging yields

$$\frac{\partial \tilde{u}^1}{\partial t} + \tilde{u}^1 \frac{\partial \tilde{u}^1}{\partial \tilde{x}^1} + \tilde{u}^3 \frac{\partial \tilde{u}^1}{\partial \tilde{x}^3} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^1} + \tilde{f}^1 \times \left[1 - \frac{(\partial z_G/\partial x)^2}{[(\partial z_G/\partial x)^2 + 1]}\right] + \frac{\partial z_G/\partial x}{[(\partial z_G/\partial x)^2 + 1]} g \frac{\theta'}{\theta_0},$$

or

$$\frac{\partial \tilde{u}^1}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^1}\left(1 - \frac{\alpha^2}{1+\alpha^2}\right) + g\frac{\alpha}{(\alpha^2+1)}\frac{\theta'}{\theta_0} + \tilde{f}^1 - \tilde{u}^1\frac{\partial \tilde{u}^1}{\partial \tilde{x}^1} - \tilde{u}^3\frac{\partial \tilde{u}^1}{\partial \tilde{x}^3}, \quad (9)$$

where $\alpha = \partial z_G/\partial x$ has been defined for convenience. For flat terrain, this relation reduces to the original Cartesian horizontal equation of motion.

A straightforward assumption for the frictional drag term $\tilde{f}^1$ in (7), for use in an analytic model in which turbulence in the along slope direction is neglected, could then be written as

$$\tilde{f}^1 = \frac{\partial}{\partial \tilde{x}^3}\left(K\frac{\partial \tilde{u}^1}{\partial \tilde{x}^3}\right). \quad (10)$$

Integrating (10) vertically between the surface and the top of the drainage flow h yields

$$\int_{\tilde{x}^3=0}^{\tilde{x}^3=h} \tilde{f}^1 d\tilde{x}^3 = K\frac{\partial \tilde{u}^1}{\partial \tilde{x}^3}\Bigg|_{\text{at } \tilde{x}^3=0}^{\text{at } \tilde{x}^3=h}, \quad (11)$$

which could be used in a layered model of drainage flow; $K[\partial\tilde{u}^1/\partial\tilde{x}^3]$ at $\tilde{x}^3 = 0$ could be approximated as $C_D(\tilde{u}^1)^2$, for example, while $K[\partial\tilde{u}^1/\partial\tilde{x}^3]$ at h could be used to represent entrainment at the top of the drainage flow. An additional advantage, therefore, of using Transformation I is that the integration in (11) is in the vertical direction rather than in the direction perpendicular to the $\tilde{x}^3$ surface.

Returning to equations (7) and (8), analytic drainage flow models can be developed using this equation of motion which are in a different form than have been used in the past for this type of meteorological problem. A different perspective to the problem should result from this representation, although as illustrated in Section 4, at least for a simple analytic drainage flow simulation, the differences obtained using the two different coordinate systems are small.

b. Transformation II

Using the definition of $\tilde{G}^{ik}$, $\partial\tilde{x}^i/\partial z$ and covariant differentiation, and Transformation II, we get

$$G^{ik} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \frac{\partial \tilde{x}^i}{\partial z} = (\sin\gamma, \cos\gamma).$$

Eq. (4) can be written in component form as

$$\frac{\partial \tilde{u}^1}{\partial t} + \tilde{u}^k\frac{\partial \tilde{u}^1}{\partial \tilde{x}^k} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^1} + g\frac{\theta'}{\theta_0}\sin\gamma + \tilde{f}^1, \quad (12)$$

$$\frac{\partial \tilde{u}^3}{\partial t} + \tilde{u}^k\frac{\partial \tilde{u}^3}{\partial \tilde{x}^k} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^3} + g\frac{\theta'}{\theta_0}\cos\gamma + \tilde{f}^3. \quad (13)$$

If a hydrostatic-type assumption is made in the $\tilde{x}^3$ direction, as suggested by Mahrt (1982) for sufficiently small slopes, (12) and (13) reduce to

$$\frac{\partial \tilde{u}^1}{\partial t} + \tilde{u}^k\frac{\partial \tilde{u}^1}{\partial \tilde{x}^k} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial \tilde{x}^1} + g\frac{\theta'}{\theta_0}\sin\gamma + \tilde{f}^1, \quad (14)$$

$$\frac{\partial p'}{\partial \tilde{x}^3} = \rho_0 g\frac{\theta'}{\theta_0}\cos\gamma. \quad (15)$$

4. Simple analytical comparison of the transformations

In order to compare quantitatively the result of the differences in the two transformations, a layer integrated momentum and thermodynamic system can be developed for a uniform slope configuration and solved analytically.

The layer integrated thermodynamic equation can be expressed in the general system by

$$\frac{\partial \bar{\theta}}{\partial t} + \tilde{u}^k \frac{\partial \bar{\theta}}{\partial \tilde{x}^k} = \bar{L}_c, \tag{16}$$

where $\bar{L}_c$ is the local warming rate for the integrated layer including both radiative and turbulent processes. Using $\bar{\theta} = \theta_0 + \theta'$ where θ_0 is a function of z only, and assuming $\partial\theta_0/\partial t = 0$, then

$$\frac{\partial \theta'}{\partial t} = -\left(\tilde{u}^1 \frac{\partial \theta_0}{\partial \tilde{x}^1} + \tilde{u}^1 \frac{\partial \theta'}{\partial \tilde{x}^1} + \tilde{u}^3 \frac{\partial \theta'}{\partial \tilde{x}^3}\right) + \bar{L}_c. \tag{17}$$

For an infinite uniform slope, $\partial\theta'/\partial\tilde{x}^1 = 0$ and mass continuity requires that $\tilde{u}^3 = 0$. Thus using the chain rule

$$\frac{\partial \theta_0}{\partial \tilde{x}^1} = \frac{\partial \theta_0}{\partial z}\frac{\partial x^3}{\partial \tilde{x}^1}, \tag{18}$$

so that (17) can be written as

$$\frac{\partial \theta'}{\partial t} = -\tilde{u}^1 \frac{\partial \theta_0}{\partial x^3}\frac{\partial x^3}{\partial \tilde{x}^1} + \bar{L}_c. \tag{19}$$

For Transformation I the generalized hydrostatic system for the uniform slope is

$$\frac{\partial \tilde{u}^1}{\partial t} = g \frac{\tan\gamma}{(\tan^2\gamma + 1)} \frac{\theta'}{\theta_0} + \tilde{f}^1, \tag{20}$$

$$\frac{\partial \theta'}{\partial t} = -\tilde{u}^1 \beta \sin\gamma + \bar{L}_c, \tag{21}$$

where $\tilde{u}^1$ and θ' represent layer quantities, $\tan\gamma = \partial z_G/\partial x$, $\sin\gamma = \partial x^3/\partial\tilde{x}^1 = \partial z/\partial\tilde{x}^1$ and $\beta = \partial\theta_0/\partial z$. Since the slope is uniform, $\partial p'/\partial\tilde{x}^1 = 0$ is assumed. Following McNider (1982) take $\partial/\partial t$ of (20) and substitute for $\partial\theta'/\partial t$ giving

$$\frac{\partial^2 \tilde{u}^1}{\partial t^2} = \frac{g}{\theta_0} \frac{\tan\gamma}{(\tan^2\gamma + 1)} (-\tilde{u}^1 \beta \sin\gamma + \bar{L}_c) + \frac{\partial \tilde{f}^1}{\partial t} \tag{22}$$

or

$$\frac{\partial^2 \tilde{u}^1}{\partial t^2} + \frac{g}{\theta_0} \beta \cos\gamma \sin^2\gamma \tilde{u}^1 - \frac{g}{\theta_0} \cos\gamma \sin\gamma \bar{L}_c = \frac{\partial \tilde{f}^1}{\partial t}. \tag{23}$$

Although the frictional case is solvable, the frictionless case can as easily permit an examination of the transformation differences. Since the slope is uniform and represents a single layer, the equation is an ordinary differential equation which for $\tilde{f}^1 = 0$ becomes

$$\frac{d^2 \tilde{u}^1}{dt^2} + \frac{g}{\theta_0} \beta \cos\gamma \sin^2\gamma \tilde{u}^1 - \frac{g}{\theta_0} \cos\gamma \sin\gamma \bar{L}_c = 0. \tag{24}$$

In a similar manner the differential equation for transformation II is

$$\frac{d^2 \hat{u}^1}{dt^2} + \frac{g}{\theta_0} \beta \sin^2\gamma \hat{u}^1 - \frac{g}{\theta_0} \sin\gamma \bar{L}_c = 0, \tag{25}$$

where a caret rather than a tilde is placed over u^1 to indicate that $\tilde{u}^1$ and $\hat{u}^1$ are different velocities. Note that the difference in the two equations [(24) and (25)] is a $\cos\gamma$ coefficient in the last two terms in (24) so that the variation in the two formulations increases for increasing slope angles. For initial conditions

$$\hat{u}^1 = \tilde{u}^1 = 0; \quad \frac{d\hat{u}^1}{dt} = \frac{d\tilde{u}^1}{dt} = 0$$

the solution for (24) [Transformation I] becomes

$$\tilde{u}^1 = \frac{\bar{L}_c}{\beta \sin\gamma}(1 - \cos\tau t) \tag{26}$$

where

$$\tau^2 = \frac{g\beta}{\theta_0} \sin^2\gamma \cos\gamma.$$

Likewise, the solution for (25) [Transformation II] is

$$\hat{u}^1 = \frac{\bar{L}_c}{\beta \sin\gamma}(1 - \cos\tau' t) \tag{27}$$

with

$$\tau'^2 = \frac{g\beta}{\theta_0} \sin^2\gamma.$$

Somewhat surprisingly the mean speed of the drainage flow is the same in both transformations with only the period of oscillation in the katabatic overshoot different. Physically this is due to the fact that both the gravitational acceleration and adiabatic deceleration terms in (24) contain the $\cos\gamma$ coefficient so that the mean effect is offset.

The change in period of the oscillation is only weakly affected by the $\cos\gamma$ term for small slopes. In fact only for slopes greater than 45° does the difference in periods between the transformations exceed approximately 30%. Thus, in spite of the variations in the development of the transformations, for most practical applications the transformations would yield identical results.

5. Conclusion

This paper has examined two coordinate representations for slope flow models—one a rotation of the coordinate axes and the other a generalized vertical coordinate transformation. Different forms of generalized hydrostatic equations were applied in the two systems in order to examine the influence of these different hydrostatic type assumptions on analytic solutions to simple idealized slope flow. Surprisingly, only the period of flow oscillation and not the mean strength of the drainage flow was changed in using the first coordinate representation instead of the second. In the coordinate rotation, acceleration was neglected perpendicular to the terrain while with the generalized vertical coordinate representation, acceleration in the σ-direction was ignored. Only for uniform slopes greater than 45°, where geophysical gravity flows probably lose their integrity anyway, does the difference in periods between the two transformations exceed 30%. Differences which may occur for nonuniform slopes, however, still need to be examined.

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