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A Terrain-Following Coordinate System – Derivation of Diagnostic Relationships

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With 1 Figure

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Summary

Generalized hydrostatic and geostrophic equations can be derived from the equations in the terrain-following framework. The generalized hydrostatic equation permits some non-hydrostatic motions (as obtained from a Cartesian framework) to remain when a non-zero slope exists. Correspondingly, the generalized geostrophic wind permits a horizontal divergent component (in addition to divergence caused by the change of Coriolis parameter with latitude) to occur when the slope angle is not zero.

1. Introduction

Since Phillips (1957) introduced the concept of a terrain-following coordinate system, this framework to represent the conservation equations has been applied extensively in meteorology. The advantage of this approach is that the ground surface coincides with a coordinate surface. Phillips applied the chain rule of calculus to transform from the Cartesian coordinate system to a terrain following coordinate framework expressed in terms of pressure divided by surface pressure.

Dutton (1976) discussed the use of tensor transformation procedures to convert from one coordinate system to another. The value of applying tensor analysis is that physical invariance is guaranteed to be preserved. Pielke and Martin (1981, 1983), Clark (1977), Pielke et al. (1985), and Pielke and Cram (1987) made use of tensor transformation techniques in deriving appropriate equations for use in meteorological models and analyses. Gal-Chen and Sommerville (1975) provided the original work in employing a z-based terrain following coordinate system to a non-hydrostatic model.

The purpose of this paper is to apply the results of tensor transformation in order to obtain new useful diagnostic relations valid on a terrain following coordinate surface.

2. Summary of Tensor Transformation Requirements

From Dutton (1976) and Pielke (1984), there are several transformation relations which will be useful when the value of this technique is discussed in Section 3. In this paper, the coordinate transformation

$$\begin{aligned}
\tilde{x}^1 &= x \\
\tilde{x}^2 &= y \\
\tilde{x}^3 &= \sigma = s[z - z_G(x,y)]/[s - z_G(x,y)] \\
x &= \tilde{x}^1 \\
y &= \tilde{x}^2 \\
z &= (\sigma/s)[s - z_G(\tilde{x}^1, \tilde{x}^2)] + z_G(\tilde{x}^1, \tilde{x}^2)
\end{aligned} \tag{1}$$

will be used to illustrate the mathematical rigor of this approach, although any single-valued functional relation between x, y and z, and a new coordinate system could be used. The terrain elevation is expressed by z_G while s is an arbitrarily selected height, usually corresponding to the top of a model. The independent variables x, y, and z are the Cartesian coordinates. This terrain-following coordinate system is used by Chang et al. (1981), Yamada (1981), Tripoli and Cotton (1983), Mahrer and Pielke (1975), Clark (1977), and others.

The vector velocity, $\vec{V}$, of course must be invariant regardless of the selected coordinate system. In the Cartesian system

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

where $\vec{i}, \vec{j}$, and $\vec{k}$ are the unit Cartesian basis vectors. In the transformed system, from Pielke (1984),

$$\vec{V} = \tilde{u}_i \vec{\eta}^i = \tilde{u}^i \vec{\tau}_j$$

$$\vec{V} = \tilde{u}_1 \vec{i} + \tilde{u}_2 \vec{j} + \tilde{u}_3 \left[\vec{i}\left(\frac{\sigma - s}{s - z_G}\right)\frac{\partial z_G}{\partial x} + \vec{j}\left(\frac{\sigma - s}{s - z_G}\right)\frac{\partial z_G}{\partial y} + \vec{k}\left(\frac{s}{s - z_G}\right)\right]$$

$$\vec{V} = \tilde{u}^1 \left[\vec{i} + \vec{k}\left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1}\right] + \tilde{u}^2\left[\vec{j} + \vec{k}\left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2}\right] + \tilde{u}^3 \vec{k}\left(\frac{s - z_G}{s}\right) \tag{2}$$

where $\vec{\eta}^i$ and $\vec{\tau}_j$ are the two sets of basis vectors that occur in a non-orthogonal coordinate system such as expressed by the transformation in (1). The velocities $\tilde{u}_i$ and $\vec{u}^i$ are the covariant and contravariant components, respectively, and are given by

$$\tilde{u}^1 = u$$

$$\tilde{u}^2 = v$$

$$\tilde{u}^3 = u\left(\frac{\sigma - s}{s - z_G}\right)\frac{\partial z_G}{\partial x} + v\left(\frac{\sigma - s}{s - z_G}\right)\frac{\partial z_G}{\partial y} + w\left(\frac{s}{s - z_G}\right)$$

$$\tilde{u}_1 = u + \left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1} w$$

$$\tilde{u}_2 = v + \left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2} w$$

$$\tilde{u}_3 = w\left(\frac{s - z_G}{s}\right). \tag{2 a}$$

Therefore, the vectors in (2) can be rewritten in terms of the Cartesian quantities as

$$\vec{V} = \left[u + \left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1} w\right]\vec{i} + \left[v + \left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2} w\right]\vec{j} + w\left(\frac{1}{s}\right)\left[\vec{i}(\sigma - s)\frac{\partial z_G}{\partial x} + \vec{j}(\sigma - s)\frac{\partial z_G}{\partial y} + \vec{k}\right]$$

$$\vec{V} = u\left[\vec{i} + \vec{k}\left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1}\right] + v\left[\vec{j} + \vec{k}\left(\frac{s - \sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2}\right] + \left[u(\sigma - s)\frac{\partial z_G}{\partial x} + v(\sigma - s)\frac{\partial z_G}{\partial y} + w(s)\right]\vec{k}\left(\frac{1}{s}\right) \tag{3}$$

The vector $\vec{V}$ presented in the Cartesian, covariant and contravariant forms are shown in Fig. 1, for a two-dimensional case.

Note that the contravariant form has a component in the vertical direction and a component parallel to the σ-surface which at $\sigma = 0$ is the ground. The covariant form, in contrast, has a component perpendicular to the σ-surface and a horizontal component.

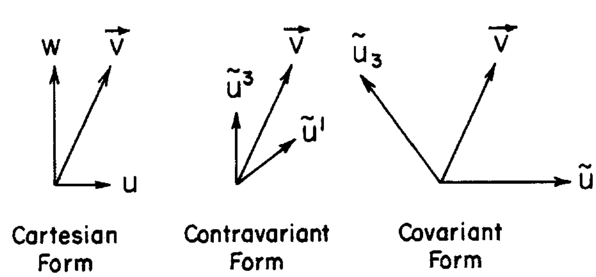

Fig. 1. Vector $\vec{V}$ as expressed in Cartesian, covariant, and contravariant components in a terrain following coordinate system. The σ-surface is parallel to $\tilde{u}^1$ and perpendicular to $\tilde{u}^3$

The consistent form of kinetic energy is $\frac{1}{2}\vec{V}\cdot\vec{V}$, which as shown by Dutton (1976) and Pielke (1984) can be calculated from

$$\vec{V}\cdot\vec{V} = \tilde{u}_i\tilde{\eta}^i\tilde{u}^j\tilde{\tau}_j = \delta^i_j\tilde{u}_i\tilde{u}^j = \tilde{u}_i\tilde{u}^i .$$

The covariant components must be multiplied by the corresponding contravariant components, i.e.,

$$\tilde{u}_i\tilde{u}^i = \tilde{u}_1\tilde{u}^1 + \tilde{u}_2\tilde{u}^2 + \tilde{u}_3\tilde{u}^3, \quad \text{or}$$

$$\begin{aligned}\tilde{u}_i\tilde{u}^i = {} & u\left[u + \left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1}w\right] \\ & + v\left[v + \left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2}w\right] \\ & + w\left[u\left(\frac{\sigma-s}{s}\right)\frac{\partial z_G}{\partial x}\right. \\ & \left. + v\left(\frac{\sigma-s}{s}\right)\frac{\partial z_G}{\partial y} + w\right] \\ = {} & u^2 + v^2 + w^2 \end{aligned} \qquad (4)$$

Equation (4) shows that $\partial/\partial \overset{1}{x} = \partial/\partial x$ and $\partial/\tilde{x}^2 = \partial/\partial y$.

Finally, the complete conservation of motion equation in the contravariant form can be written in general form (from Pielke, 1984, pg. 110) for the coordinate transformation given by (1) as

$$\begin{aligned}\frac{\partial \tilde{u}^1}{\partial t} = {} & -\tilde{u}^j\frac{\partial \tilde{u}^1}{\partial \tilde{x}^j} - \theta\frac{\partial \pi}{\partial \tilde{x}^1} \\ & + \theta\frac{\sigma-s}{s-z_G}\frac{\partial z_G}{\partial x}\frac{\partial \pi}{\partial \tilde{x}^3} - \hat{f}\tilde{u}^3 + f\tilde{u}^2 \end{aligned} \qquad (5\,a)$$

$$\begin{aligned}\frac{\partial \tilde{u}^2}{\partial t} = {} & -\tilde{u}^j\frac{\partial \tilde{u}^2}{\partial \tilde{x}^j} - \theta\frac{\partial \pi}{\partial \tilde{x}^2} \\ & + \theta\frac{\sigma-s}{(s-z_G)}\frac{\partial z_G}{\partial y}\frac{\partial \pi}{\partial \tilde{x}^3} - f\tilde{u}^1 \end{aligned} \qquad (5\,b)$$

$$\begin{aligned}\frac{\partial \tilde{u}^3}{\partial t} = {} & -\tilde{u}^j\frac{\partial \tilde{u}^3}{\partial \tilde{x}^j} - \frac{1}{(s-z_G)} \\ & \left[(s-\sigma)\frac{\partial^2 z_G}{\partial \tilde{x}^{1^2}}(\tilde{u}^1)^2\right. \\ & + (s-\sigma)\frac{\partial^2 z_G}{\partial \tilde{x}^{2^2}}(\tilde{u}^2)^2 \\ & + 2(s-\sigma)\frac{\partial^2 z_G}{\partial \tilde{x}^1\partial \tilde{x}^2}\tilde{u}^1\tilde{u}^2 \\ & \left. - 2\frac{\partial z_G}{\partial \tilde{x}^1}\tilde{u}^1\tilde{u}^3 - 2\frac{\partial z_G}{\partial \tilde{x}^2}\tilde{u}^2\tilde{u}^3\right] \\ & - \theta\left\{\frac{\partial z_G}{\partial x}\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial \pi}{\partial \tilde{x}^1}\right. \\ & + \frac{\partial z_G}{\partial y}\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial \pi}{\partial \tilde{x}^2} \\ & + \left[\left(\left(\frac{\partial z_G}{\partial x}\right)\left(\frac{\sigma-s}{s-z_G}\right)\right)^2\right. \\ & + \left(\left(\frac{\partial z_G}{\partial y}\right)\left(\frac{\sigma-s}{s-z_G}\right)\right)^2 \\ & \left.\left. + \left(\frac{s}{s-z_G}\right)^2\right]\frac{\partial \pi}{\partial \tilde{x}^3}\right\} - \frac{s}{s-z_G}g \end{aligned} \qquad (5\,c)$$

where, to reduce the notational complexity the Coriolis term was left out of the $\tilde{u}^3$ equation[1]. In (5 a) and (5 b) $f = 2\Omega\sin\varphi$; $\hat{f} = 2\Omega\cos\varphi$; with $\Omega = 2\pi$/day and φ is latitude. From the analysis summarized earlier in this section, equation (5 a) and (5 b) are specified parallel to σ-surfaces while (5 c) is in the vertical axis along a σ-coordinate.

3. Diagnostic Relations

A useful form of (5 c) can be derived if it is assumed that vertical accelerations are small compared to the remaining terms, which yields

$$\begin{aligned}0 = {} & -\theta\left\{\frac{\partial z_G}{\partial x}\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial \pi}{\partial \tilde{x}^1}\right. \\ & + \frac{\partial z_G}{\partial y}\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial \pi}{\partial \tilde{x}^2} \\ & + \left[\left(\left(\frac{\partial z_G}{\partial x}\right)\left(\frac{\sigma-s}{s-z_G}\right)\right)^2\right. \\ & + \left(\left(\frac{\partial z_G}{\partial y}\right)\left(\frac{\sigma-s}{s-z_G}\right)\right)^2 \\ & \left.\left. + \left(\frac{s}{s-z_G}\right)^2\right]\frac{\partial \pi}{\partial \tilde{x}^3}\right\} - \frac{s}{s-z_G}g \end{aligned} \qquad (6)$$

[1] One can manipulate equations (5 a–c) as performed by Clark (1977) and Clark (1988, personal communication), such that explicit prognostic equations for $\tilde{u}^1$, $\tilde{u}^2$ and w (the cartesian vertical velocity) are obtained. This rearrangement makes use of the contravariant velocity component definitions in equation 2 a. These equations can be written in a flux form, which is advantageous for computational accuracy, as pointed out by Clark. The mathematical equivalence of the Christoffel symbols still occur in his equations, however, appearing in his diagnostic pressure equation which is derived using equations 5 a through 5 c.

Rearranging (6) to solve for $\frac{\partial \pi}{\partial \tilde{x}^3}$ produces

$$\frac{\partial \pi}{\partial \tilde{x}^3} = -\left(\frac{s}{s - z_G}\frac{g}{\theta} + \frac{\partial z_G}{\partial x}\left(\frac{\sigma - s}{s - z_G}\right)\frac{\partial \pi}{\partial \tilde{x}^1} + \frac{\partial z_G}{\partial y}\left(\frac{\sigma - s}{s - z_G}\right)\frac{\partial \pi}{\partial \tilde{x}^2}\right) \Bigg/ \left[\left(\left(\frac{\partial z_G}{\partial x}\left(\frac{\sigma - s}{s - z_G}\right)\right)\right)^2 + \left(\left(\frac{\partial z_G}{\partial y}\left(\frac{\sigma - s}{s - z_G}\right)\right)\right)^2 + \left(\frac{s}{s - z_G}\right)^2\right] . \quad (7)$$

Equation (7) is a generalized hydrostatic equation since accelerations are neglected in the σ-direction but permitted in the σ-parallel orientation, thereby retaining some non-hydrostatic motion when referred back to the Cartesian hydrostatic equation[2].

When slope angles are small (7) reduces to

$$\frac{\partial \pi}{\partial \tilde{x}^3} = -\left(\frac{s - z_G}{s}\right)\frac{g}{\theta} \quad (8)$$

which is the form generally applied to represent the hydrostatic assumption in atmospheric models with a terrain following coordinate system. Equation (8) is the shallow slope generalized hydrostatic approximation. Physick (1986), however, did apply the complete form given by (7) in his simulation of the flow in the Grand Canyon.

A generalized geostrophic wind can be derived from (5 a) and (5 b) where a balance between the pressure gradient force and the Coriolis term is assumed, and equation (8) is used to represent $\partial \pi / \partial \tilde{x}^3$, i.e.,

$$\tilde{u}_g^2 = \frac{1}{f}\left[\theta \frac{\partial \pi}{\partial \tilde{x}^1} - g\frac{\sigma - s}{s}\frac{\partial z_G}{\partial x}\right]$$

$$\tilde{u}_g^1 = -\frac{1}{f}\left[\theta \frac{\partial \pi}{\partial \tilde{x}^2} - g\frac{\sigma - s}{s}\frac{\partial z_G}{\partial y}\right]$$

(where, for simplicity, the $\tilde{f}\tilde{u}^3$ term was ignored). The generalized gesotrophic components, $\tilde{u}_g^1$ and $\tilde{u}_g^2$ are, of course, parallel to σ-surfaces. If these components are zero

$$\frac{\partial \pi}{\partial \tilde{x}^1} = \frac{fg}{\theta}\left(\frac{\sigma - s}{s}\right)\frac{\partial z_G}{\partial x};$$

$$\frac{\partial \pi}{\partial \tilde{x}^2} = -\frac{fg}{\theta}\left(\frac{\sigma - s}{s}\right)\frac{\partial z_G}{\partial y}$$

is required $\left(\text{i.e.,} \frac{\partial \pi}{\partial \tilde{x}^1} = \frac{\sigma - s}{s - z_G}\frac{\partial z_G}{\partial x}\frac{\partial \pi}{\partial \tilde{x}^3}; \frac{\partial \pi}{\partial \tilde{x}^2} = \frac{\sigma - s}{s - z_G}\frac{\partial z_G}{\partial y}\frac{\partial \pi}{\partial \tilde{x}^3}\right)$.

Since $\tilde{u}_g^1 \vec{\tau}_1$ and $\tilde{u}_g^2 \vec{\tau}_2$ are the components on σ-surfaces, a horizontal component of the generalized geostrophic wind can be derived from

$$u_g = \tilde{u}_g^1 \vec{\tau}_1 \cdot \vec{i}; \quad v_g = \tilde{u}_g^2 \vec{\tau}_2 \cdot \vec{j}$$

which, using the form of $\vec{\tau}_j$ and $\tilde{u}^j$ in (2) and (2 a) yields

$$u_g = u; \quad v_g = v$$

so that the horizontal geostrophic wind derived from the terrain-following coordinate system is the same as would be derived in a Cartesian framework.

The main results of this summary analysis is that:

i) a generalized hydrostatic equation can be derived which retains some non-hydrostatic motions when referred back to the Cartesian coordinate system. Pielke et al. (1985) discusses this approach as applied to drainage flow models.

ii) a generalized geostrophic wind can be derived which is parallel to the σ-surfaces (e.g., Sangster, 1960, 1987). Pielke and Cram (1987) used such an approach to obtain a horizontal geostrophic wind which is a more reasonable estimate of geostrophic pressure gradients in elevated, irregular terrain, than is the standard approach of reducing pressure to sea level through the elevated terrain.

4. Conclusion

Generalized hydrostatic and geostrophic equations can be derived from the equations of motion expressed in the terrain-following coordinate framework. The generalized hydrostatic equation permits some non-hydrostatic motions (as ob-

[2] It is important to recognize that while the Cartesian hydrostatic equation represents a balance of forces between the vertical pressure gradient and gravitational forces, the generalized hydrostatic equation as defined here retains those vertical accelerations which are parallel to σ-surfaces.

tained from a Cartesian framework) to remain when a non-zero slope exists. Correspondingly, the generalized geostrophic wind permits a horizontal divergent component (in addition to divergence caused by the change of the Coriolis parameter with latitude) to occur when the slope angle is not zero.

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