Chaos, Strange Attractors, and Weather

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Abstract

Some of the basic principles of the theory of dynamical systems are presented, introducing the reader to the concepts of chaos theory and strange attractors and their implications in meteorology. New numerical techniques to analyze weather data according to the above theory are also presented.

1. Introduction

Simplicity and regularity are associated with predictability. For example, because the orbit of the earth is simple and regular we can always predict when astronomical winter will come. On the other hand, complexity and irregularity are almost synonymous with unpredictability. The atmosphere, being so complex and irregular, is rather unpredictable.

Those who try to explain the world we live in always hoped that in the realm of the complexity and irregularity observed in nature, simplicity would be found behind everything, and finally unpredictable events would become predictable. That complexity and irregularity exist in nature is obvious. We only need to look around us to realize that practically everything is random in appearance. Or is it? Clouds, like many other structures in nature, come in an infinite number of shapes. Every cloud is different, yet everybody will recognize a cloud. Clouds, though complex and irregular, must on the whole possess a uniqueness that distinguishes them from other structures in nature. The question remains: is their irregularity completely random or is there some order behind their irregularity?

Over the last decades physicists, astronomers, biologists, and scientists from many other disciplines have developed a new way of looking at complexity in nature. This way has been termed *chaos* theory.

Chaos theory, which mathematically defines randomness generated by simple deterministic dynamical systems, allows us to see order in processes that were thought to be completely random. (Apparently, the founders of chaos theory had a very good sense of humor, since chaos is the Greek word for the complete absence of order.) It is the purpose of this paper to introduce the reader to some chaos theory concepts and some implications of chaos theory in weather and climate.

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2. Simple examples and definitions from the theory of dynamical systems

In the preceding paragraph the term "dynamical systems" was used. What is a dynamical system? In simple terms a *dynamical system* is a system whose evolution from some initial state (which we know) can be described by a set of rules. These rules may be conveniently expressed as mathematical equations. The evolution of such a system is best described by the so-called "state space." An example of a simple dynamical system, a pendulum, and its state space, is given below.

Consider a pendulum that is allowed to swing back and forth from some initial state, as shown in figure 1a. The initial state can be completely described by the velocity, v, and the position of the pendulum. The position of the pendulum at any time can be given by the angle x. Under such an arrangement, Newtonian physics provides the equations (rules) that describe the system's evolution from the initial state.

Let us assume that the pendulum starts at position 1. At position 1 its initial state will be $x = x_1$, and velocity $v = 0$. The pendulum is then let free. As it moves towards point 0, its speed increases due to gravity acceleration. After a while (position 2), the pendulum will be closer to point 0 and will have a higher speed. Once the pendulum crosses point 0 its speed decreases, since now gravity acts in a direction opposite to its motion. At some point (position 3), the pendulum's speed will become zero again. Immediately the pendulum will begin to swing back. After it crosses point 0 it will once again attain, at some point, a zero speed (position 4). Because there is always some friction, however, the points at which the speed becomes zero (to the right and left of point 0) are not fixed but are found closer and closer to point 0. Finally, the pendulum will come to rest at point 0.

Apparently, the time evolution of the pendulum can be completely described by two variables, namely velocity and angle. These two variables define the

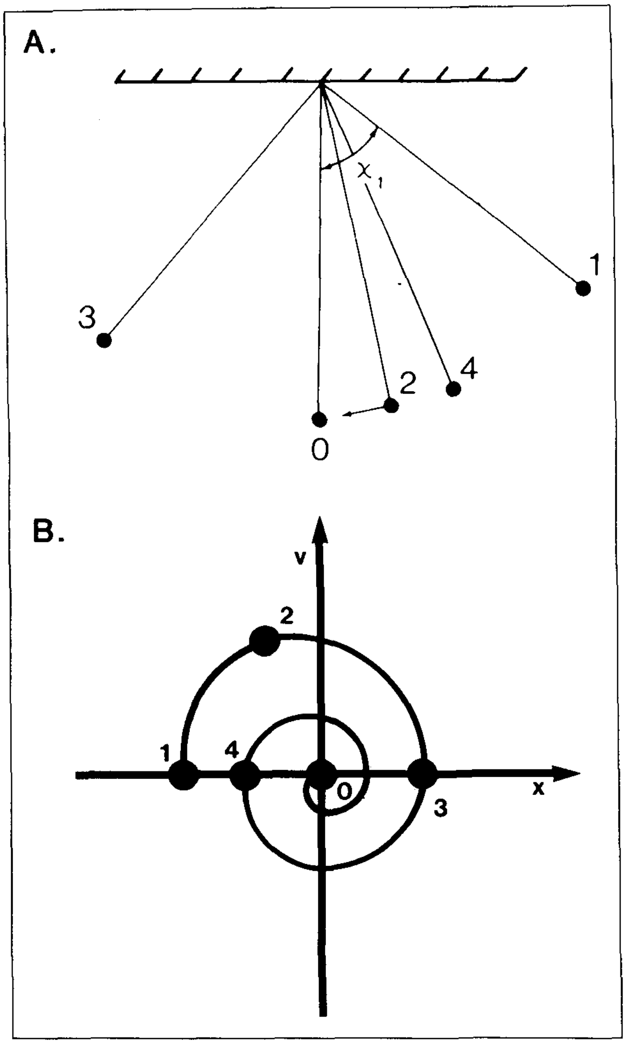

FIG. 1.(a) A *dynamical system* is a system whose evolution from some initial state can be determined by some rules. In the above figure the motion of the pendulum can be completely described by the laws of physics if its initial position and velocity are known. (b) An example of a dynamical system whose coordinates are the velocity and the angle of the pendulum. As the pendulum swings back and forth it follows a trajectory in the state space which converges to a fixed point, or attractor of the dynamical system.

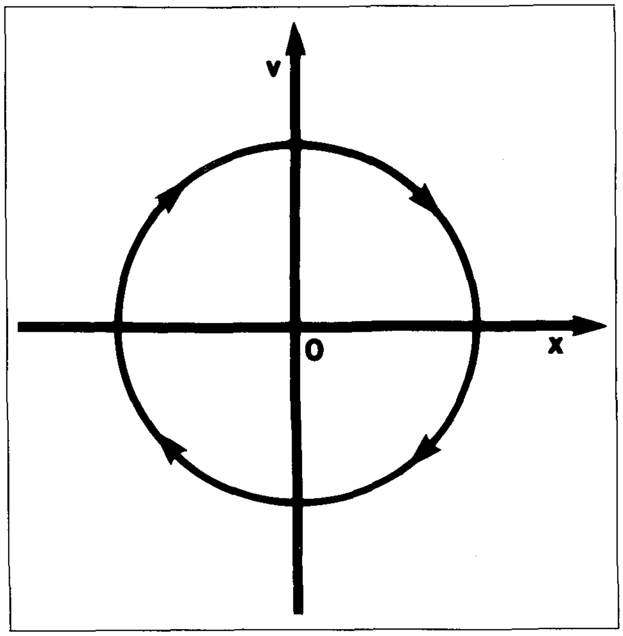

FIG. 2. Another form of an attractor is the limit cycle. In this case all trajectories are attracted by the limit cycle, which represents a period evolution. The pendulum of a grandfather clock is a system that possesses a limit cycle as an attractor. Another familiar system with a limit cycle as its attractor is the heart.

coordinates of the state space. If one plots the velocity (v) as a function of the angle (x) of the pendulum, for the times corresponding to positions 1, 2, 3, 4, one will arrive at figure 1b. Each point represents the state of the system at a given moment, therefore a trajectory that connects all points gives a visualization of the evolution of the system. As shown, the trajectory converges, that is stops, at point 0. As a matter of fact, any other trajectory that corresponds to an evolution of this dynamical system from a different initial state (velocity and position) will converge at point 0 (i.e. no matter what the initial state, the pendulum will always come to rest at point 0). The point 0 in the state space is called an *attractor.* It "attracts" all the trajectories in the state space. Apparently, the behavior of the dynamical system in question can be completely understood. Long-term predictability is guaranteed. The pendulum will always come to rest at point 0. Point attractors therefore correspond to systems that reach a state of no motion.

So far we have discussed only one form of attractor (a point). The next simplest form of attractor is the limit cycle (figure 2). A limit cycle in the state space indicates a periodic motion. An example of a system whose attractor is a limit cycle is the grandfather clock, in which the loss of kinetic energy due to friction is compensated mechanically via a mainspring. No matter how the pendulum clock is set swinging, a perpetual, periodic motion will be achieved. This periodic motion manifests itself in the state space as a limit cycle. Again, in the case of systems that have a limit cycle as an attractor, long-term predictability is guaranteed.

Another form of attractor is the torus. The torus looks like the surface of a doughnut (figure 3). In this case, all the trajectories in the state space are attracted to and remain on the surface. Systems that possess a torus as an attractor are quasi-periodic. In a quasi-periodic evolution a periodic motion is modulated by a second motion, itself periodic, but with another frequency. The combination of frequencies will produce a time series whose regularity is not clear. The power spectrum, however, should consist of sharp peaks at each of the basic frequencies with all its other prominent features being combinations

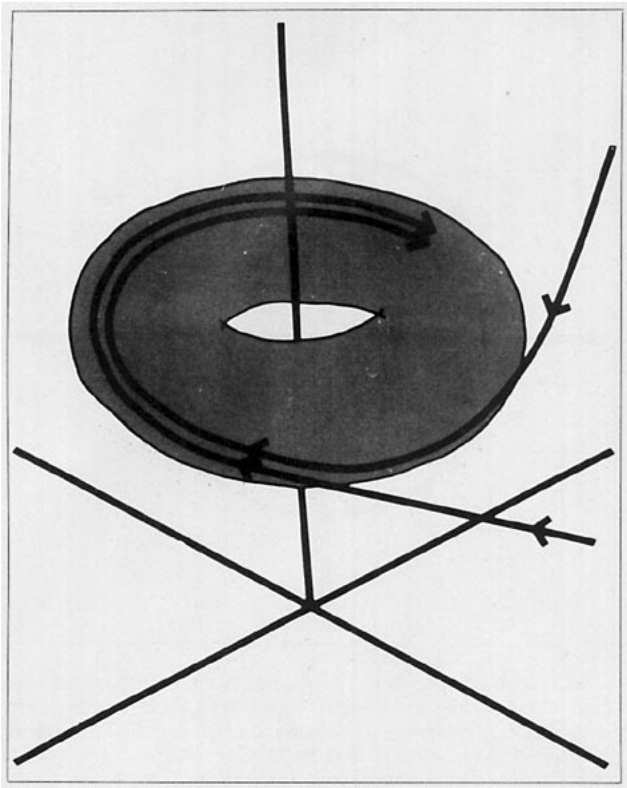

FIG. 3. Another form of an attractor is the torus. In this case the evolution of the corresponding dynamical system from any initial condition will follow a trajectory in the state space that will eventually be attracted and remain forever on the torus. The most important characteristic of a system that exhibits such an attractor is that usually two initially nearby trajectories on the attractor remain nearby forever.

of the basic frequencies. Geometrically, a quasi-periodic trajectory fills the surface of a torus, in the appropriate state space (Thompson and Steward 1986). An important characteristic of such an attractor is that when the two frequencies have no common divisor, any two trajectories which represent the evolution of the system from different initial conditions, and which are close to each other when they approach the attracting surface, will remain close to each other forever (see figure 3). This characteristic can be translated as follows. The two points in the state space where the trajectories enter the attractor can be two measurements (initial states) which differ by some amount. Since these trajectories remain close to each other, the states of the system at a later time are going to differ to the same extent that they differed initially. Thus, if we know the evolution of such a system from an initial condition, we can predict accurately the evolution of the system from some other initial condition. Again, in this case long-term predictability is guaranteed.

The above mentioned forms of attractors are "well-behaved" attractors and usually correspond to systems whose evolution is predictable. Often they are

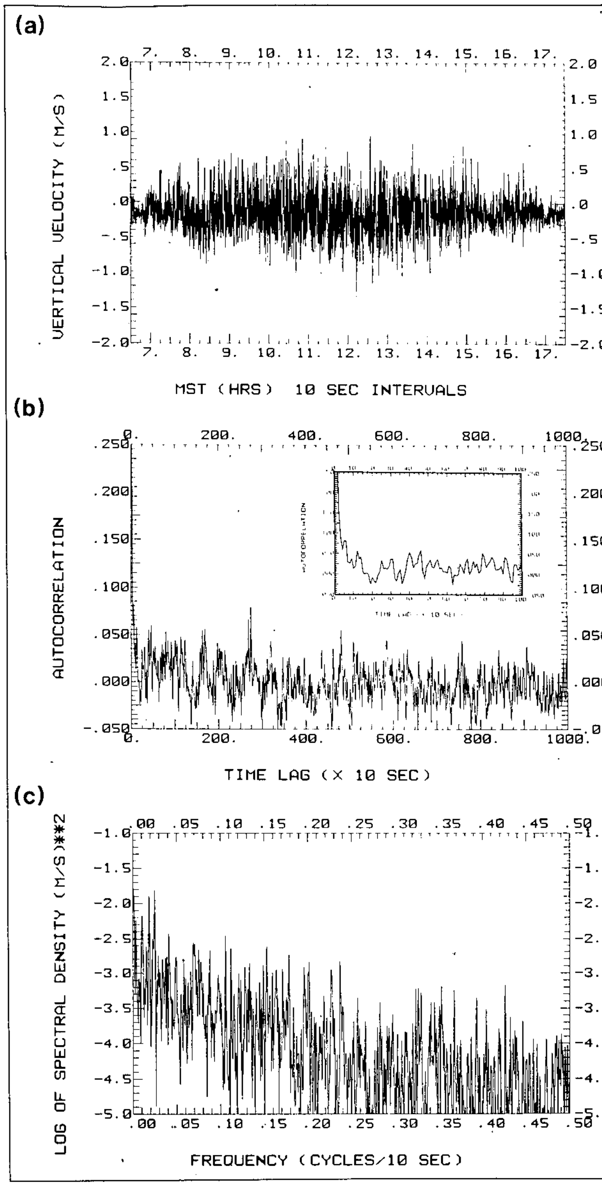

FIG. 4.(a) The data represent 10-second averages of the vertical wind velocity over 11 hours. The data were recorded from 6:30 to 17:30 MST (Mountain Standard Time) (12:30 to 23:30 UTC) on 26 September 1986 at Boulder, Colorado. At about 6:30 (12:30) the sun rises. The air close to the ground is heated and rises creating strong convection. Positive values indicate updrafts and negative values indicate downdrafts.
(b) The autocorrelation function for the above data. The inset graph is a magnification of the region close to the origin.
(c) The logarithm of the spectral density as a function of the frequency for the above data. The spectra shows various peaks on a background of continuous frequency spectrum, suggesting a non-periodic evolution.

called non-chaotic attractors. In mathematical terms, the above mentioned attractors are smooth topological submanifolds of the available state space. These attractors are, therefore, characterized by an integer dimension that is equal to the topological dimension of the submanifold in the state space. A very important characteristic of these attractors is that the long-term evolution of the systems they describe is not sensitive to initial conditions.

3. Strange attractors

When one observes the spectra of turbulent motion, one realizes that there is motion at all frequencies with no preferred frequencies (see data in figure 4). This broad-band structure of the spectrum indicates that the motion is nonperiodic (or strictly speaking is periodic with an infinite period). Could such a nonperiodic motion be due to a simple dynamical system? Let us assume that the answer to this question is yes. In such a case the trajectory in the state space would be nonperiodic (never repeat itself) and never cross itself (since once a system returns to a state it was in some time in the past it then has to follow the same path). Thus the trajectory should be of infinite length but confined to a finite area in the state space. This can only be the case if the attractor is not a topological manifold but rather is a fractal set (see figure 5 and table 1).

The first such system was discovered in 1963 by Edward Lorenz (Lorenz 1963). This system, described by the following three differential equations, gives an approximate description of a horizontal fluid layer heated from below. The fluid at the bottom gets warmer and rises, creating convection. For a choice of the constants that correspond to sufficient heating, the convection may take place in an irregular and turbulent manner:

$$dx/dt = -ax + ay$$
$$dy/dt = -xz + bx - y$$
$$dz/dt = -xy - cz$$

where x is proportional to the intensity of the convective motion, y is proportional to the horizontal temperature variation, z is proportional to the vertical temperature variation, and a, b, and c are constants. Figure 6a depicts the path of a trajectory in the state space (x, y, z). The Lorenz attractor itself does not look like the well-behaved attractors previously described. The trajectory is deterministic (since it is the result of the solution of the above system of equations), but is strictly nonperiodic. The trajectory loops to the left and then to the right irregularly. Extensive studies have shown that the fine structure of the Lorenz attractor is made up of infinetely nested layers (infinite area) that occupy zero volume. One may think of it as a Cantor-like set in a higher dimension. Its fractal (Hausdorff-Besicovitch) dimension has been estimated to be about 2.06 (see for example Grassberger and Procaccia 1983a).

The fractal nature of an attractor does not only im-

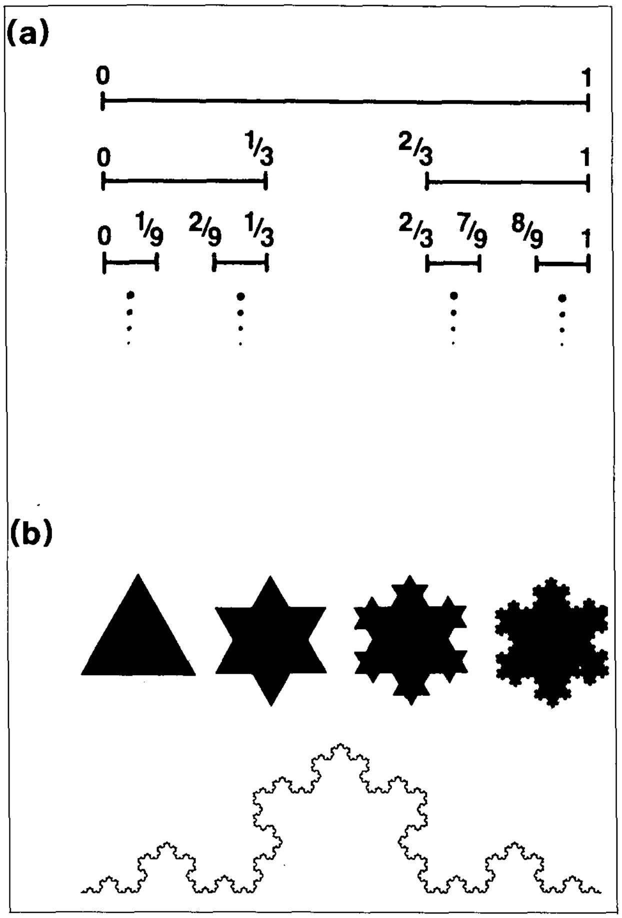

FIG. 5. Fractals (Mandelbrot, 1983) are sets that are not topological. For sets that are topological the Hausdorff-Besicovitch dimension is an integer (0 for points, 1 for any curve, 2 for any surface etc.). For sets that are fractal their Hausdorff-Besicovitch dimension is not an integer but a real number. Because of that, fractal sets have properties that topological manifolds do not have: a) the Cantor set begins with a line of length one; then the middle third is removed; then the middle third of all the remaining intervals is removed and so on. The Cantor set or Cantor "dust" is the number of points that remain. The total length of all intervals removed is $1/3 + 2(1/3)^2 + 4(1/3)^3 + 8(1/3)^4 + \ldots\ldots = 1$. Thus, the length remaining must be zero. Therefore, in the Cantor set the number of points is obviously infinite but their total length is zero. The Hausdorff-Besicovitch dimension of this set (see section 4 for ways of measuring this dimension) is 0.6309. It is definitely greater than the topological dimension of a "dust" of points which is zero. b) the Koch curve begins with an equalateral triangle with sides of length one; then at the middle of each side a new equalateral triangle with sides of length one-third is added; and so on. The length of the constructed boundary is $3 \times 4/3 \times 4/3 \times 4/3 \times \ldots\ldots = \infty$. However, that boundary occupies no area at all and it encloses a finite area which is less than the area of a circle drawn around the original triangle. The Hausdorff-Besicovitch dimension of the Koch curve is 1.2618 (higher than the topological dimension of any curve which is equal to one). Often the Hausdorff-Besicovitch dimension is referred to as the fractal dimension. Extentions of the above to higher dimensions should be obvious. Such mathematical curiosities, abstract as they seem, have now found a place in the study of dynamical systems. The Koch curve has been reproduced from Mandelbrot (1983) after permission from the author. More on the application of fractals in meteorology can be found in Lovejoy and Schertzer (1986).

TABLE 1. Defining an infinite number of generalized dimensions.

- Hausdorff-Besicovitch (or fractal, or capacity) dimension, D_0

$$D_0 = \lim_{l \to 0} \frac{\log N(l)}{\log (l^{-1})}$$

where $N(l)$ is the number of n-dimensional cubes of length l needed to cover a set embedded in an n-dimensional Euclidean space.

- Information dimension, D_1

$$D_1 = \lim_{l \to 0} \frac{1}{\log l} \sum_{i=1}^{N(l)} P_i \log P_i$$

where P_i is the probability of a point from a long time series to fall in cube i of the covering.

- Correlation dimension, D_2

$$D_2 = \lim_{r \to 0} \frac{\log C(r)}{\log r}$$

where $C(r)$ is the correlation function defining the number of pairs of points in the set with distance less than r. The definition of the correlation dimension D_2 can be extended to consider higher order correlation functions that define the number of triplets, of quatruplets and of n-tuplets of points. This way an infinite number of generalized dimensions D_3, D_4 . . . . . D_n can be obtained. In general it can be proved that $D_0 > D_1 > D_2 > \ldots > D_n$ where the inequality is replaced by the equality only in special cases (Hentschel and Procaccia 1983).

TABLE 1. Given a fractal set there exist an infinite number of different (and relevant) generalized dimensions (or exponents) that characterize that set. The fact that fractal measures lead to all those different exponents was first noted in Mandelbrot (1974). Because of that, today many times we talk in terms of *multifractals* were a hierarchy of fractal dimensions can be defined. A complete knowledge of the set of generalized dimensions or exponents is equivalent to a complete characterization of a fractal set or a strange attractor. For some of the consequences and applications of multifractals in meteorology see Lovejoy et al. (1987) and references therein.

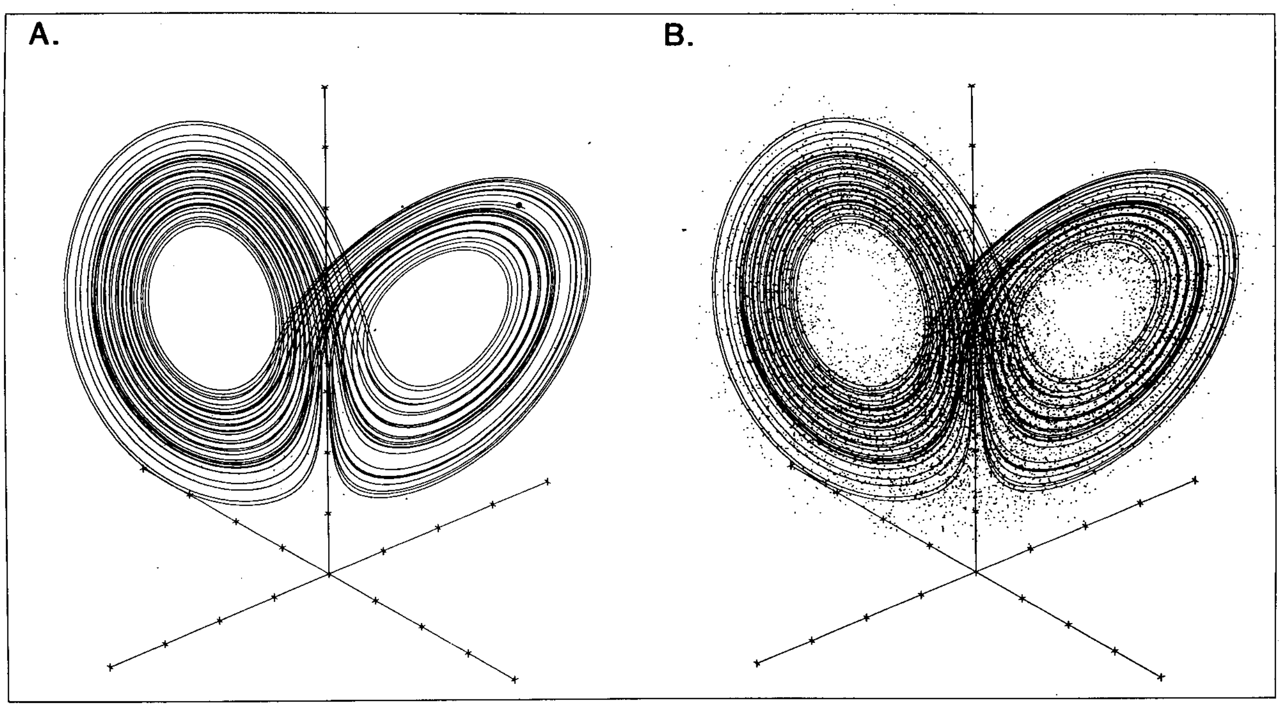

FIG. 6.(a) An example of a strange attractor with implications in the weather forecasting problem. This structure in the state space represents the attractor of a fluid flow which travels over heated surface. All trajectories (which will represent the evolution of that system for different initial conditions) will eventually converge and remain on that structure. However, any two initially nearby trajectories in the attractor do not remain nearby, but diverge. (Figure courtesy of Dr. James Crutchfield).
(b) The effect of the divergence of initially nearby trajectories in the attractor: The dot in FIG. 6(a) represents 10,000 measurements (initial conditions) that are so very close to each other that they are practically indistinguishable. If we allow each of these states to evolve according to the rules, because their trajectories diverge irregularly, after a while their states can be practically anywhere. (Figure courtesy of Dr. James Crutchfield).

ply nonperiodic orbits. It also causes nearby trajectories to diverge. As with all attractors, trajectories that are initiated from different initial conditions soon reach the attracting submanifold, but two nearby trajectories do not stay close to each other (as was the case with the torus). They soon diverge and follow totally different paths in the attractor.

The divergence means that the evolution of the system from two slightly different initial conditions will be completely different, as may be seen in figures 6a and 6b. The dot in figure 6a represents 10,000 initial conditions that are so close to each other in the attractor that they are indistinguishable. They may be viewed as 10,000 initial situations that differ only slightly from each other. If we allow these initial conditions to evolve according to the rules (equations) that describe the system, we see (figure 6b) that after some time the 10,000 dots can be anywhere in the attractor. In other words, the state of the system after some time can be anything despite the fact that the initial conditions were very close to each other. Apparently, the evolution of the system is very sensitive to initial conditions. In this case we say that the system has generated randomness. We can now see that there exist systems that, even though they can be described by simple deterministic rules, can generate randomness. Randomness generated this way has been termed *chaos*. These systems are called chaotic dynamical systems and their attractors are often called strange or chaotic attractors.

The implications of such findings are profound. If one knows exactly the initial conditions, one can follow the trajectory that corresponds to the evolution of the system from those initial conditions and basically predict the evolution forever. The problem, however, is that we cannot have a perfect knowledge of initial conditions. Our instruments can only measure approximately the various parameters (temperature, pressure, etc.) that will be used as initial conditions. There will always be some deviation of the measured from the actual initial conditions. They may be very close to each other, but they will not be the same. In such a case, even if we completely know the physical laws that govern our system, due to the nature of the underlying attractor the state of the system at a later time can be totally different from the one predicted. Simply, due to the nature of the system, initial errors are amplified and therefore prediction is limited. Recently, ideas from the theory of chaotic dynamical systems have been applied to simple models that describe climatic fluctuations and transitions between ice ages and today's climate (Nicolis 1987; Tsonis and Elsner 1988b). These studies attribute the broad-band structure of the spectrum of observed climatic data to the presence of nonperiodic chaotic attractors. The attractors are very sensitive to the initial conditions, in accordance with the intrinsic unpredictability of the climatic system (Lorenz 1984).

4. The search for attractors in weather and climate

One very important consequence of knowing the Hausdorff-Besicovitch dimension of an attractor is that the dimensionality of an attractor, whether fractal or not, indicates the minimum number of variables present in the evolution of the corresponding dynamical system (in other words the attractor must be embedded in a state space of at least its dimension). Therefore, the determination of the Hausdorff-Besicovitch dimension (or for that matter of any other generalized dimension) of an attractor sets a number of constraints that should be satisfied by a model used to predict the evolution of a system.

If the mathematical description of a dynamical system is given, the number of variables is known and the generation of the state space and of the attractor is straightforward.

If the mathematical formulation of a system is not available, the state space can be replaced by the so-called phase space. The phase space may be produced using a single record of some observable variable $x(t)$ from that system (Packard et al. 1980; Reulle 1981; Takens 1981). The physics behind such an approach is that a single record from a dynamical system is the outcome of all interacting variables and thus information about the dynamics of that system should, in principle, be included in any observable variable. The mathematical procedure used to prove the existence and nature (chaotic or not) is as follows. It is assumed that variables present in the evolution of the system in question satisfy a set of n first-order differential equations:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \ldots\ldots\ldots x_n)\\ \dot{x}_2 &= f_2(x_1, x_2, \ldots\ldots\ldots x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, x_2 \ldots\ldots\ldots x_n)\end{aligned}$$

where one dot indicates the first derivative with respect to time. In such a case the coordinates of the state space are $(x_1, x_2 \ldots\ldots\ldots x_n)$. The above system of differential equations can be reduced to one highly nonlinear differential equation of n^{th} order. This can be achieved by successive differentiation of one of the equations describing the system (equation 1). In this way, one may obtain an n^{th} order differential equation.

$$x_1^{(n)} = f(x_1, \dot{x}_1, \ldots\ldots\ldots, x^{(n-1)})$$

and replace the coordinates of the state space with $(x_1, \dot{x}_1, \ddot{x}_1, \ldots\ldots\ldots x_1^{(n-1)})$ or $(\dot{x}_1, \ddot{x}_1, \ldots\ldots\ldots x_1^{(n)})$ without any loss of information about the evolution or the dynamics of the system. Ruelle (1981) suggested that instead of a continuous variable $x(t)$ and its derivatives, a discrete time series $x(t)$ and its successive shifts by a delay parameter τ should be sufficient to identify attractors in the time evolution of a single variable (one should think of the shifting operation on a discrete-time series as analogous to differentiation of a continuous-time series). Thus, given an observable $x(t)$, one can generate the complete state vector $\mathbf{X}(t)$ by using $x(t + \tau)$ as the first coordinate, $x(t + 2\tau)$ as the second coordinate, and $x(t + n\tau)$ as the last coordinate. This way we can define the coordinates of the phase space, which should approximate the dynamics of the system from which the observable $x(t)$ was sampled (or in other words the unknown state space). The parameter n is often referred to as the *embedding dimension.* For an n-dimensional phase space, a "cloud" or a set of points will be generated. The Hausdorff-Besicovitch dimension of this set can be estimated by covering the set by n-dimensional cubes of side length l and determining the number of cubes $N(l)$ needed to cover the set in the limit $l \rightarrow 0$ (Mandelbrot 1983). This is the so called box-counting algorithm and if this number scales as

$$N(l) \propto l^{-d} \qquad l \longrightarrow 0$$

(equation 2) then the scaling exponent d is an estimation of the Hausdorff-Besicovitch dimension for that n. In a log $N(l)$, log l plot the exponent d can be estimated by the slope of a straight line. Using the state vector $\mathbf{X}(t)$ we can test equation 2 for increasing values of n. If the original time series is random, then $d = n$ for any n (a random process embedded in a n-dimensional space always fills that space). If, however, the value of d becomes independent of n (that is, reaches a saturation value D_0) it means that the system represented by the time series has some structure and should possess an attractor whose Hausdroff-Besicovitch dimension is equal to D_0. The above procedure for estimating D_0 is a consequence of the fact that the actual number of variables present in the evolution of the system is not known and thus we do not know a priori what n should be. We must, therefore, vary n until we "tune" to a structure which becomes invariant in higher embedding dimensions (an indication that extra variables are not needed to explain the dynamics of the system in question).

The above numerical approach to estimate the dimension of an attractor from a time series is, however, very limited. The reason for that is that an enormous number of points on the attractor is required to make sure that a given area in the phase space is indeed empty and not simply visited rarely. It has been documented (Froehling et al. 1981 and Greenside et al. 1982) that a box-counting approach is not feasible for phase space dimensions greater than two.

An alternative approach which is much more applicable has been developed by Grassberger and Procaccia (1983a and 1983b). This approach again generates in an n-dimensional phase space a cloud of points. But instead of covering the set with hypercubes, it finds the number of pairs $N(r,n)$ with distances less than a distance r. In this case, if for significantly small r, we find that

$$N(r,n) \propto r^{d_2}$$

(equation 3) we call the scaling exponent d_2 the correlation dimension of the attractor for that n. We then test equation 3 for increasing values of n and check as previously for a saturation value D_2, which will be an estimation of the correlation dimension of the attractor. It should be mentioned at this point that τ can be small, but care should be taken not to include in the sums pairs whose time separation is less than the correlation time. The correlation dimension D_2 is less than the Hausdorff-Besicovitch (or fractal) dimension D_0 and actually measures the spatial correlation of the points that lie on the attractor. For a random time series there will be no such spatial correlation in any embedding dimension and thus no saturation will be observed in the exponent d_2. The above approach still requires a large number of points (especially for high embedding dimensions), but at least it is more feasible than the box-counting method. Recently, a new algorithm was proposed which promises to be even more applicable (Badii and Politi 1985). This algorithm is called the nearest neighbor algorithm. The algorithm calculates the distance between each point and its nearest neighbor, δ_i where $i = 1, \ldots\ldots\ldots N$ and N is the total number of points in the phase space. The γ-moment of this distance is related to a dimension-function $D(\gamma)$ according to the relation in equation 4:

$$\left[\frac{1}{N}\sum_{i=1}^{N} \delta_i^{\gamma}\right]^{1/\gamma} \propto N^{[1/D(\gamma)]}$$

The dimension function is related to the generalized dimensions, D_q, via the expression $D(\gamma = (1 - q)Dq) = Dq$ for $q = 0, 1, 2, \ldots\ldots\ldots n$. Thus, by applying equation 4 for choices of γ, information about all the exponents that characterize the attractor can be inferred. This technique leads, according to Badii and Politi, to rather robust estimates of the dimension for smaller N than the pair-counting algorithm. However, it is a relatively new technique and has not yet been applied extensively to weather data.

The search for attractors in weather and climate is driven by the desire to investigate the observed complexity in the atmosphere. The theory of chaos provides us with new tools to do just that. An exact mathematical formulation of atmospheric processes has not yet been developed, therefore, observable weather variables are considered in the search for attractors in weather and climate. One such application involves the data shown in figure 4 and has been reported by Tsonis and Elsner (1988a). The data represent 10-second averages of the vertical wind velocity recorded 10 meters above the ground over an 11-hour period. The decorrelation time was defined as the lag time at which the autocorrelation falls below a value of 0.10. This value can be inferred from figure 4 as approximately 20 seconds. From these data the state vector $\mathbf{X}(t)$ was generated and the phase space was produced for embedding dimensions two and higher using $\tau = 10$ sec. For each embedding dimension the number of pairs $N(r,n)$ with distances less than r is then found as a function or r. Then the logarithm $N(r,n)$ is plotted against the logarithm of r. Figure 7 shows these plots for selected embedding dimensions. From this information and for each embedding dimension, the scaling region is determined and its slope is calculated by fitting a straight line in that region. The slope of the straight line gives the scaling exponent d_2 in equation 3. More details on the above application, discussion about the scaling behavior of a particular data set, and the possible problems associated with fitting a straight line to intervals of r for which r is too large or too small, can be found in Tsonis and Elsner (1988a) and Essex et al. (1987). Figure 8 shows the scaling exponent as a function of the embedding dimension, together with a plot representing the time series as a random sample of the same size as the vertical velocity data set. From that figure and for the wind data it was estimated that $D_2 = 7.3$. Note that no saturation for exponent d_2 is observed for the random sample. It can be concluded that the system represented by the vertical wind velocity (the atmosphere over very short time scales in this case) possesses an attractor. Since the dimensionality of this attractor is noninteger, the attracting submanifold is a fractal set (i.e. the attractor is strange). Such a finding provides us with the minimum number of degrees of freedom (differential equations) that are needed to reproduce the dynamics of weather on very short time scales.

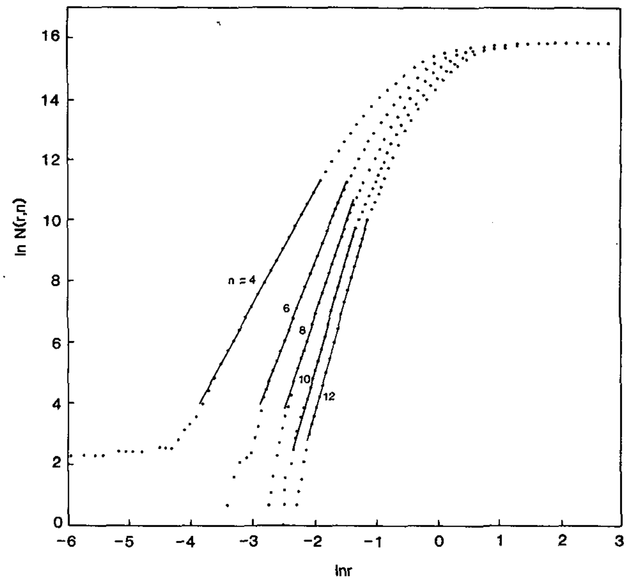

FIG. 7. Plot of *ln* $N(r,n)$ against *lnr* for embedding dimensions, $n = 4, 6, 8, 10, 12$. Note the convergence of slopes as n increases.

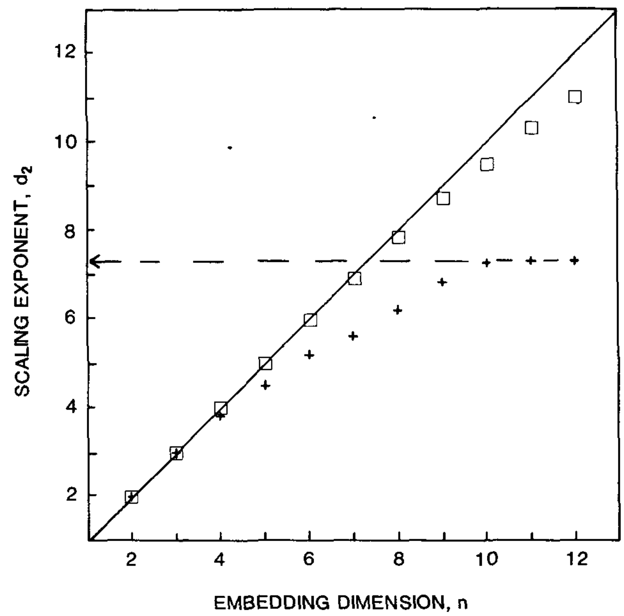

FIG. 8. Scaling exponent, d_2, as a function of the embedding dimension, n. Crosses correspond to the wind velocity data and squares to a random sample of the same size as the wind data. Note the saturation of the scaling exponent observed for the wind data while there is no saturation for the random set. From this Figure it is estimated that $D_2 = 7.3$.

Other notable related studies include the analysis of Nicolis and Nicolis (1984), who were the first to apply the above ideas to climatic studies. They used single-variable values of oxygen isotope records of deep sea cores spanning the past million years (see also Fraedrich 1987). These data are related to global temperature fluctuations during that time interval. They reported a dimension for the climatic attractor equal to 3.1. Fraedrich (1986) applied the same analysis over a period of 15 years using daily pressure data, and Essex et al. (1987) applied the analysis over a period of 40 years using daily geopotential data. Both groups reported a dimension of the weather attractor between 6 and 7. Recently, as we described above, Tsonis and Elsner (1988a) extended that analysis to very short time scales. All the above studies have made use of the pair-counting algorithm and, technically speaking, may all suffer to an extent from using fewer than required data points (Smith 1987 and 1988). That will cause some uncertainty about the estimated value of the dimension of the attractor. Nevertheless, very long records are, unfortunately, not available most of the time. New and more efficient algorithms will probably take care of such limitations. At this point, although the existence or not of attractors in weather is still an open question, it is significant that the empirical studies indicate that low-dimensional attractors may be present in weather and climate. The fact that the inferred dimensions seem to be different for different time scales may indicate that the attractors (and thus predictability) are different and a function of the time scale. On the other hand, it may be that when considering a certain time scale we are only looking at a certain part of a grand attractor. Both possibilities are very exciting. It is anticipated that research in this area will provide many clues about the predictability and interaction of different time scales.

5. Conclusions

Chaos theory has opened new horizons in science and is already considered by many to be the most important discovery in the twentieth century after relativity and quantum mechanics.

Many systems in nature are chaotic. The developments in the study of chaotic dynamical systems have suggested that nature imposes limits on prediction. At the same time, however, it has been realized that the very existence of the attractors implies that randomness is restricted to the attractors. The atmosphere may be chaotic, but its evolution is confined to a specific area in the state space that is occupied by the attractor. No states outside this area are allowed. The winds associated with a high pressure system, for example, can never blow counterclockwise.

The theory of chaotic dynamical systems has improved our understanding of the behavior of the atmosphere. At the same time, even though it has provided an excuse for the unpredictability of weather, the theory of dynamical systems is slowly shaping our way of investigating the weather and its prediction. For example, it may very well be that generalizations based on the study of specific cases (which may never happen exactly again) can no longer be appropriate.

Together with some pessimism, the study of chaotic dynamical systems provides some optimism. We may never be able to predict the weather exactly, but improvements in weather forecasting are feasible if we improve the completeness and accuracy with which we measure the initial condition of the atmosphere, and if we understand predictability at different time scales.

The impact of chaos theory has already been felt in many areas of science. If we give it a chance its impact will be felt in the atmospheric sciences as well. After all, chaos was essentially discovered by a meteorologist.

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