# Butterfly Effects of the First and Second Kinds: New Insights Revealed by High-dimensional Lorenz Models

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**New Insights Revealed by High-dimensional Lorenz Models**

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**Abstract.** Over the span of 50 years, the pioneering study of Lorenz using a three-dimensional (Lorenz) model (3DLM) in 1963 and follow-up studies in 1969 and 1972 have changed our view on the predictability of weather and climate by revealing the so-called butterfly effect. Although Lorenz's '63 and '72 studies emphasized nonlinear dynamics, researchers often apply a "simple" conceptual model that contains a monotonic positive feedback instead of time-varying (positive or negative) nonlinear feedback in order to understand the characteristics of nonlinear solutions within the 3DLM. In this study, we: (1) define butterfly effects of the first and second kinds in order to indicate the sensitive dependence of solutions on initial conditions, and the hypothesized enabling role of tiny perturbations in producing an organized large-scale system (e.g., a tornado), respectively; (2) illustrate important but overlooked features (i.e., the boundedness and recurrence of solutions within the 3DLM); (3) present examples to illustrate common misunderstandings regarding butterfly effects and explain the fundamental differences between the two kinds of butterfly effects; (4) illustrate the fundamental role of nonlinearity in creating oscillatory components with incommensurate frequencies, transferring energy across scales, and providing negative or positive feedbacks; (5) discuss various types of solutions (e.g., chaotic, linearly unstable, and/or nonlinear oscillatory solutions) in Lorenz models; and (6) propose that the entirety of weather is a superset that consists of both chaotic and non-chaotic processes. Depending on the time-varying collective impact of heating, dissipations, and nonlinearity, specific weather systems may appear on a chaotic or non-chaotic orbit for a finite period of time.

# 1 Introduction

The discovery of a "butterfly effect" within a three-dimensional nonlinear Lorenz model (3DLM, Lorenz 1963) changed our view on the predictability of weather and climate. While a long history exists regarding when and how the term butterfly effect was first introduced, as suggested in the book entitled *The Essence*

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*of Chaos* by Lorenz in 1993, the term appeared to obtain noticeable attention after Lorenz's studies in 1963 and 1972 (Lorenz 1963, 1972). By conducting a comprehensive literature review, we illustrate that the current meaning of the butterfly effect and its major characteristics is not exactly the same as proposed by Lorenz in his studies of Lorenz (1963, 1972). Additionally, although Lorenz (1963, 1972) emphasized nonlinear dynamics (i.e., chaotic dynamics), researchers often apply a "simple" conceptual model that contains a monotonic positive feedback instead of time-varying (positive or negative) nonlinear feedback in order to understand the characteristics of nonlinear solutions within Lorenz models. Outstanding questions and issues remain from Lorenz's original studies (Lorenz 1963, 1972), for example: (1) we still need to properly interpret Lorenz's studies in order to improve our understanding of butterfly effect(s), (2) we need to understand the nature of the relationship between butterfly effects in these two studies, and (3) we need to know what role nonlinear processes play within Lorenz models.

In this study, we provide a brief review and a summary of our recent studies using newly developed high-dimensional LMs (Shen 2014, 2015a, b, 2016, 2017; Shen et al., 2018b). We first define and discuss butterfly effects of the first and second kinds (BE1 and BE2) in Lorenz's studies (1963, 1972). Within the BE1, we illustrate important, but overlooked features of the butterfly pattern solution. We then analyze the Lorenz (1969) model in order to show the necessary conditions for BE2. As suggested by the recent studies of Palmer et al. (2014) and Durran and Gingrich (2014), as well as Rotunno and Snyder (2008), Lorenz (1969) may be the first attempt at proposing a multiscale system for "revealing" features that are "associated with" the BE2.

**The BE1 in Lorenz (1963)**

Based on the pioneering Lorenz study in 1963, the original meaning (or "definition") of the butterfly effect is a "sensitive dependence of solutions on initial conditions (ICs)", suggesting that a tiny change in an IC can produce a very different time evolution of a solution for three variables (X, Y, Z). These variables represent the amplitudes of either stream functions or temperatures and describe an orbit or trajectory within the phase space. The dimension of the phase space is equal to the number of variables. For example, three, five, and seven dimensional Lorenz models (3D, 5D, and 7DLM) include three, five, and seven variables, respectively (e.g., Shen 2014, 2016, 2017). The term "dimension" is conventionally used for ordinary differential equations (ODEs, e.g., Hirsch et al. 2013; Thompson and Stewart 2002). In this study, the 5D and 7DLM are referred to as high-dimensional Lorenz models. In existing literature, they are often referred to as high-order Lorenz models (e.g., Moon et al. 2017).

The sensitive dependence of solutions on ICs has been illustrated using the divergence of two, initial, nearby trajectories within the phase space of the 3DLM. For example, using the model with typical parameters (e.g., the Rayleigh parameter $r = 28$, to be discussed in Eqs. (A1)-(A3) in Appendix A), Figures 1(a)-1(c) display a very different time evolution for two solution orbits whose starting points are very close to one another. In addition to the divergence of nearby trajectories, the solutions or orbits are bounded. Partly due to its geometric pattern within the phase space (e.g., Figure 2a of this study or Figure 2 of Lorenz 1993), this phenomenon was originally used to define the butterfly effect (i.e., the BE1). A butterfly pattern with a finite size and varying curvatures within the phase space (as shown in Figure 2a) also qualitatively suggests important features of solution boundedness. For such a system, the error (or divergence) of two orbits should be bounded (e.g., Figure 1d).

The average separation speed (i.e., an average divergence) of nearby trajectories has been quantitatively measured using the Lyapunov exponent (LE, Wolf et al. 1985; Zeng et al. 1991, 1993). A positive LE suggests an exponential rate in the separation of two nearby trajectories over a long period. The 3DLM and high-dimensional LMs have a positive LE within a range of parameters. Therefore, both a positive LE and solution boundedness are used to define a chaotic system. Although not correct, since the LE is computed over an infinite period of time, researchers often misinterpret the divergence of two trajectories associated with a positive LE within the 3DLM as continuing over time and lasting forever. Due to solution boundedness, a trajectory should recurve within the phase space. Therefore, time-varying (local) growth rates along a chaotic orbit are observed (e.g., Zeng et al. 1993) and may become negative, as indicated by a negative finite time LE (e.g., Figure 7 of Nese 1989; Figure 1 of Eckhardt and Yao 1993; p. 397 of Ding and Li 2007; Figure 3 of Bailey 2011). When a trajectory returns back to the neighborhood of a previously visited state, "recurrence" is defined. Recurrence may be viewed as a generalization of "periodicity" that braces quasi-periodicity and chaos (Thompson and Stewart 2002) and may appear as a result of solution divergence (with a positive LE), boundedness, and recurvature. Specifically, recurrence is an essential ingredient[1] for the irregular oscillations of a strange attractor. A chaotic system has been stated to possess an infinite number of unstable periodic solutions (Mirus and Sprott, 1999).

In reality, the term butterfly flaps should indicate very small amplitudes of flaps at tiny temporal (for a short-term period) and spatial scales (with a small size). In particular, tiny scales are the scales of real butterflies or sea gulls (p. 15 of Lorenz 1993). However, as a result of the limited degree of spatial scale interactions with three modes, it can be said that the 3DLM reveals the complexities of the solution only within the temporal space. While the word butterfly for the BE1 indeed represents the pattern of the solution that consists of all possible outcomes of the system, the BE1 emphasizes the various time sequence of outcomes, a very unique feature of the system. A tiny perturbation in the initial conditions at one of the selected spatial scales (e.g., a perturbation, $\epsilon$, in the state variable Y in Figure 1) is often viewed as a butterfly flap. Such a perturbation can only modify the time sequence of various events within the

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[1] The characteristics of solution recurrence can be found in the definitions of chaos of Devaney (1989): (1) sensitivity to initial conditions; (2) topological transitivity; and (3) dense periodic points.

deterministic Lorenz system (e.g., Lorenz 1993). Note that such tiny perturbations within the 3DLM do not hold realistic amplitudes or the temporal or spatial scales required to represent real butterflies. If proper rescaling is applied to represent butterflies within a 3DLM that only has three scales, a flap by a butterfly can change its journey but cannot make changes on other events that have scales different from the three pre-selected scales. Stated alternatively, the solution of the 3DLM is sensitive to an interior change (in ICs). Note that a change in system parameters is an external change. Therefore, while the 3DLM can be used to define the BE1, it is not a proper model for addressing the BE2 that requires scale interactions at multiple spatial scales (i.e., many modes are required). Table 1 summarizes major features of the 3DLM and the BE1.

**The BE2 in Lorenz (1972)**

Here, we use the BE2 to indicate the enabling role of a tiny perturbation in producing an organized large-scale system (e.g., a tornado). Note that the definition of BE2 requires interactions at various (physical) spatial scales (e.g., the butterfly and the tornado) and energy transferring across scales (e.g., intermediate scales between the scales of a butterfly and a tornado). The BE2 was originally discussed by Lorenz (1972) who raised the following three, and only three, questions:

1. *Predictability; Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?*
2. *In more technical language, is the behavior of the atmosphere unstable with respect to perturbations of small amplitude?*
3. *How can we determine whether the atmosphere is unstable?*

Simply speaking, Lorenz addressed the first question (regarding the BE2) by answering the second and third questions. He specifically linked the BE2 with the release of instability by the atmosphere and the capability of transferring energy by tiny perturbations. In recognition of the limited size of a butterfly and the limited performance of numerical methods in transferring the butterfly's influence across different regions, as discussed in Section 2, Lorenz stated:

- *One hypothesis, unconfirmed, is that the influence of a butterfly's wings will spread in turbulent air, but not in calm air;*
- *We must therefore leave our original question (i.e., the first question) unanswered for a few more years, even while affirming our faith in the instability of the atmosphere (i.e., the second and third questions).*

The above suggests that the impact of a butterfly heavily depends on the instability of the atmosphere[2]. However, the presentation of Lorenz (1972) did not provide a mathematical model for addressing the source of instability and the

[2] It should be noted that the term "chaos" was introduced into nonlinear dynamics by the study of Li and Yorke in 1975. Therefore, detailed differences between instability and chaos have been discussed since that time. See an example in Appendix.

transference of perturbations across scales. A butterfly's ability in creating an organized weather system was also not addressed.

As suggested by recent studies, (some) major features for the BE2 were first addressed using the Lorenz (1969) model. Lorenz (1969) used a simple partial differential equation (PDE) with a nonlinear advection term that describes the evolution of vorticity. A PDE is an equation that involves partial derivatives in both time and space. A common approach is to convert a PDE into a set of ODEs in order to understand fundamental physical processes. By applying a linearization method (Hartman 1963) using a basic state that possesses a realistic spectrum (i.e., realistic amplitudes over a range of spatial scales), Lorenz (1969) transformed the PDE into a set of linear ODEs in order to describe the time change of "perturbations" (i.e., departures from the basic state) at different spatial scales. An initial condition of the perturbation may be viewed as a butterfly's flap, while the basic state represents the atmosphere. Major findings are briefly discussed below, while details of the mathematical approaches and numerical experiments are provided in the Supplemental Materials[3] of Shen et al. (2018a, in preparation).

The spread and growth of initial perturbations at various scales were discussed. The dependence of growth rates on spatial scales (i.e., wavelengths) was illustrated and growth rates were used to estimate predictability. As a result of smaller growth rates, larger-scale systems were presumed to have better predictability. A predictability limit of several weeks[4] was suggested (e.g., Lewis, 2005; p. R139 of Palmer et al. 2014). Major findings in Lorenz (1969) have been supported in studies conducted near the time of Lorenz's 1969 publication (e.g., Leith and Kraichnan 1972) and in recent studies using more sophisticated PDEs (e.g., Rotunno and Snyder 2008; Durran and Gingrich 2014). We provide a summary of the Lorenz (1969) model in Table 1. After introducing the concept of "multiscale" using our high-dimensional LMs, we will provide detailed comments as to whether the 1969 model is good for addressing BE2 (as well as BE1) in Section 2.

Up to this point, we have defined the BE1 and BE2 and discussed other important but overlooked features in chaotic solutions. In Section 2, we present examples in order to discuss common misunderstandings regarding the butterfly effect and identify a nonlinear feedback loop (NFL) and its extensions in order to address the source of recurrence and negative (positive) feedbacks for suppressing (enhancing) chaos using high-dimensional LMs (Shen 2014, 2015a,b, 2016; Shen and Faghih-Naini 2017; Moon et al. 2017; Shen et al., 2018b; Faghih-Naini and Shen 2018). We then discuss the relationship between BE1 and BE2. Concluding remarks are provided at the end.

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[3] Twenty-three years after Lorenz (1972), Prof. Lorenz addressed predictability issues by proposing new models in Lorenz (1996, 2005) that are nonlinear chaotic systems with many modes but not derived from physics-based PDEs.

[4] We agree with Prof. Arakawa that the predictability limit is not necessarily a fixed number" (Lewis, 2005).

## 2 New Insights Revealed by High-dimensional LMs

In this section, we: (1) discuss major features of the 3DLM (Lorenz, 1963), (2) provide a summary of recent studies using generalized high-dimensional LMs that were extended based on the 3DLM, and (3) apply our findings to improve our understandings of the Lorenz (1969) model.

### 2.1. Overlooked Features within the 3DLM

To facilitate discussions, we present Eqs. (A1-A3) from the original Lorenz model (1963) and discuss one popular, but inaccurate, analogy for chaos in Appendix A. We use this example to illustrate the important features of chaotic solutions that should include divergence and boundedness, which depend on the competitive or collective impact of three kinds of processes (e.g., nonlinear processes and linear heating and dissipative processes). The strength of heating is measured by the normalized Rayleigh parameter (r). Depending on whether the Rayleigh parameter is below or above a threshold of 24.74, two types solutions (i.e., steady state and chaotic solutions) are generally discussed. However, based on the relative strength of the above processes, we show two sets of overlooked solutions within the 3DLM, as well as unique characteristics within high-dimensional LMs, and apply them in order to refine the current view of weather being chaotic.

The first set includes both steady-state and chaotic solutions. While chaotic solutions appear when the Rayleigh parameter exceeds the critical value ($r_c$) of 24.74, they may co-exist with steady-state solutions over a small range of r (i.e., $24.06 < r < 24.74$) (e.g., p. 333 of Ott 2002; p. 242 of Drazin 1992). As discussed later, such coexistence also occurs in high-dimensional LMs. In comparison, the second set of overlooked solutions appear when the Rayleigh parameter becomes larger (say, $r > R_c$; $R_c = 313$, Sparrow 1982; Shimizu 1979; Strogatz 2015). These solutions are isolated and closed. For a stable orbit, nearby orbits approach it, indicating its isolated nature. As a result, these outcomes are referred to as limit cycle solutions. One interesting characteristic for a limit cycle is that its orbit is solely determined by the system and independent of the ICs. Therefore, an initial error may play a role in triggering this type of solution. An important message for the appearance of limit cycle solutions at larger Rayleigh parameters is that chaotic solutions only occur over a finite range of Rayleigh parameters[5]. In the past, the fundamental dynamics of the limit cycle have been illustrated using a grandfather clock: periodicity is maintained by both a driving force (e.g., the

[5] Similar findings for the dependence of various solutions (i.e., chaotic and limit cycle solutions) on the strength of heating were also reported using a two-layer, quasi-geostrophic model that describes the finite-amplitude evolution of a single baroclinic wave by Pedlosky and Frenzen (1980).

restoring force of the spring in the clock) and dissipation (e.g., friction in the air). Within the 3DLM, while the isolated nature requires dissipation, a "closed" nature with periodicity is achieved by nonlinearity alone or the competition of heating and nonlinearity within the non-dissipative model (e.g., Shen 2018). Below, associations of limit cycle solutions with nonlinear terms are briefly discussed so we can provide information regarding the source of recurrence within the 3DLM[6].

**2.2 The Linear Uncoupled Geometric Model**

The linearized system of Eqs. (A1-A3) with respect to the trivial critical point, which only has linear forcing (e.g., heating and buoyancy) and dissipation terms, but no nonlinear term, produces either a bounded steady solution or an unbounded unstable solution. While the geometric model of Guckenheimer and Williams (1979) consists of three linear uncoupled ODEs, a proof for dynamic equivalence between the geometric model and the 3DLM by Tucker (2002) revealed the existence of the Lorenz strange attractor. Tucker's study suggests the important role of a saddle point at the original in producing a sensitive dependence of solutions on ICs. However, as discussed in Shen et al. (2018b), an additional assumption of "return conditions" within the geometric model requires justification.

**2.3 The Non-dissipative 3DLM and the "Limiting" Equations**

By examining the nonlinear 3DLM, Shen (2018) recently illustrated the fundamental role of nonlinear terms in producing periodicity. We first analyzed the term for the nonlinear advection of temperature within the Rayleigh-Benard convection (RBC) equations in order to identify the nonlinear terms (i.e., XY and –XZ in Eqs. (A2) and (A3)) as a pair of downscaling and upscaling processes and, thus, defined them as a nonlinear feedback loop (NFL, Shen 2014). Assuming no dissipation, the 3DLM can be simplified so that it only contains nonlinear and heating processes. Within the nonlinear, nondissipative 3DLM (3D-NLM), we have shown that the NFL acts as a nonlinear restoring force to produce oscillatory solutions. We have also shown that the 3D-NLM with $r = 0$ is, indeed, the same as the *"Limiting"* Equations of Sparrow (1982, Eq. (2) on p. 133), a simplified model used to reveal oscillatory solutions under conditions of large Rayleigh parameters (i.e., limit cycle solutions, to be specific).

**2.4 Linearized Non-dissipative High-dimensional Lorenz Models**

Recently, the role of the extended NFL has been examined using non-dissipative high-dimensional LMs (e.g., Faghih-Naini and Shen 2018; Shen et al., 2018c).

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[6] While a geometric model was proposed for illustrating the characteristics of the Lorenz strange attractor (Guckenheimer and Williams, 1979), the model does not include nonlinear terms for recurrence.

By linearizing the 3D-NLM and high-dimensional, non-dissipative LMs (Shen 2018; Shen and Faghih-Naini, 2017; Shen et al., 2018c), we showed that an extension of the NFL with two additional modes for temperature can introduce an additional frequency that is incommensurate with existing frequencies. For example, Faghih-Naini and Shen (2018) found that as compared to the linearized 3D-NLM that produces a periodic solution with one frequency, the linearized 5D-NLM produces a quasi-periodic solution with two incommensurate frequencies (e.g., Figure C of Faghih-Naini and Shen (2018)). Using a special type of generalized NLM, Shen et al. (2018c) showed that the number of incommensurate frequencies is equal to (M-3)/2 + 1, where M represents the number of modes and is an odd number. Shen et al. (2018c) also found that a composite motion in higher-dimensional, non-dissipative LMs may look more complicated within the temporal space. Figure 3 indicates that the linearized 5D-NLM (7D-NLM) is mathematically identical to systems with two (three) springs and two (three) masses.

Since a linearized system is mathematically simpler than its nonlinear version, it is effective for revealing energy transferring across spatial scales. However, a linearized system is only good for examining the evolution of a solution near the non-trivial critical point over a short period of time. Specifically, the linearized system cannot reproduce chaotic features (e.g., irregular oscillations between two butterfly's wings) that require a nonlinear system. Below, using nonlinear dissipative versions, we illustrate the collective impact of the extended NFL, as well as additional dissipation and heating terms.

### 2.5 Nonlinear, Dissipative High-dimensional Lorenz Models

In the following, we illustrate that when dissipations are added back to the system, the impact is two-fold: (1) a damping of high-frequency modes and (2) negative nonlinear feedback. The first feature is discussed using a linear stability analysis of the dissipative 3DLM, 5DLM, and 7DLM (e.g., Shen 2014, 2016). The analysis indicates that one, two, and three pairs, respectively, of complex eigenvalues appear near the non-trivial critical point, producing oscillatory components that may represent a growing oscillation, a decaying oscillation, or a simple oscillation. In general, due to stronger dissipations at higher wavenumber modes, the higher-frequency mode has a larger decay rate. Since oscillatory components with larger decay rates dissipate quickly, we suggest that strong dissipations do indeed reduce the complexities of solutions associated with multiple incommensurate frequencies within high-dimensional LMs.

Regarding the second feature, we have previously shown that the collective impacts of the NFL (and its extensions) and the dissipative terms have the ability to provide a negative feedback for stabilizing a system that requires a larger critical value for the Rayleigh parameter ($r_c$) for the onset of chaos. For example, the $r_c$ for the 5DLM, 7DLM, and 9DLM are 42.9, 116.9, and 679.8, respectively, as compared to a $r_c$ of 24.74 for the 3DLM (e.g., Figure 2; Table 1 of Shen 2016; Shen et al., 2018b). Similar to the 3DLM, chaotic solutions still appear over a finite range of Rayleigh parameters within the 5DLM, 7DLM, and 9DLM, as well as for higher-order LMs (e.g., Moon et al. 2017), requiring larger Rayleigh parameters for limit cycle solutions as compared to the 3DLM. Negative feedback can be found within the so-called Lorenz-Stenflo system that extends the 3DLM with one additional ODE containing one additional mode that takes rotation into consideration (e.g., Xavier and Rech 2010; Park et al. 2015, 2016). With the new 9DLM and the generalized Lorenz model (Shen et al., 2018b,c), we can reveal the aggregated negative feedback based on the successive negative feedback from smaller-scale modes.

### 2.6 Aggregated Negative Feedback in a Generalized Lorenz Model (GLM)

Based on recent studies where we extended the NFL, we derived the generalized LM (GLM) in order to reveal the aggregated negative feedback that leads to a larger effective dissipation in higher dimensional LMs (Shen et al. 2018 b, c). The GLM produces consistent results that a larger r is required for the onset of chaos or a limit cycle solution within higher-dimensional LMs. More importantly, the GLM with 9 modes (e.g., the 9DLM) was used to effectively reveal the co-existence of two types of solutions. A linear stability analysis within the 9DLM suggests that the origin is still a saddle point but that non-trivial critical points are stable for any Rayleigh parameter with $\sigma = 10$ and b = 8/3. The appearance of stable, non-trivial critical points indicates a stronger aggregated negative nonlinear feedback within the 9DLM, as compared to the 5DLM and 7DLM, see details in Shen et al. (2018b). Unique features within the 9DLM allow two special sets of solutions: (1) the coexistence of chaotic and steady-state orbits with moderate Rayleigh parameters (679.8 < r < 1600) and (2) the coexistence of limit cycle/torus orbits and spiral sinks with large Rayleigh parameters (r >= 1600). The first type shares properties similar to that of the 3DLM, where co-existence only appears for 24.06 < r < 24.74. In comparison, co-existence appears over a wider range of Rayleigh parameters within the 9DLM. Additionally, the second type of co-existence has never been documented in studies using Lorenz models[7]. As a result of (1) the coexistence of chaotic and non-chaotic orbits[8] and (2) the dependence of various types of solutions on the heating parameter, we suggest that, contrary to the traditional view that weather is chaotic, weather is, in fact, a superset that consists of both chaotic and non-chaotic processes.

### 2.7 Positive Feedback in High-Dimensional Lorenz Models

In contrast to negative feedbacks, positive feedbacks can also be identified in high-dimensional LMs. While the two additional, high-wavenumber modes of the 5DLM can provide negative nonlinear feedbacks for stabilizing solutions, as compared to the 3DLM, a third new mode within the 6DLM introduces an additional heating term that can destabilize solutions, as compared to the 5DLM.

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[7] A preliminary simulation was presented in the $12^{th}$ slide of Shen et al. (2018a).
[8] The coexistence of chaotic and quasi-periodic orbits has been recently documented in a modified Lorenz system by Saiki et al. (2017).

The 6DLM (Shen 2015b) requires a slightly smaller $r_c$ for the onset of chaos as compared to the 5DLM, while both have larger $r_c$s than the 3DLM. The impact associated with the additional heating term is referred to as a positive feedback. Comparison of the 7DLM with a different 9DLM can reveal an additional positive feedback (Shen 2016, 2017).

### 2.8 A "Rough" Analogy with a Tree

Based on the above discussions, to illustrate the role of the NFL in a system, here, we use a tree as an analogy. The NFL in a system is viewed as the main trunk of a tree and its extensions as growth of the main trunk and branches. A bigger tree with a larger and stronger main trunk and branches may possess greater interconnections and is, thus, more stable (i.e., less vulnerable under windy conditions) as compared to a smaller tree. Additionally, asynchronous vibrations of the leaves may act as dissipations for stabilizing the branches and the tree (e.g., James et al. 2006). Therefore, a larger-scale sophisticated modeling system with a "healthy" interconnection of the NFL and its extension could be more stable as compared to a smaller-scale simplified modeling system. However, depending on the balance of the main trunk, various branches, and leaves, a growing tree may not always increase its stability. One reasonable "hypothesis" may be drawn here: the probability of destroying a tree by a butterfly's flap should be zero.

### 2.9 An Analysis of the Lorenz 1969 Model

The aforementioned analyses are applied in the following in order to determine whether the Lorenz (1969) model is a good tool for revealing BE1 and BE2. The Lorenz 1969 model contains many scales but it is linear. The linearized advection term within the 1969 model is responsible for energy spreading, while the basic state with a realistic spectrum serves as an energy source for various growth rates on different scales. However, unstable solutions within the linear system contain constant growth rates and, therefore, are fundamentally different from chaotic solutions. The corresponding solution grows at a constant exponential rate and is, therefore, not bounded. Growing perturbations are not allowed to provide feedback to the basic state whose changes should impact the availability of instability. An assumption of linearity has been addressed in a recent study by Durran and Gingrich (2014) who added a very simple nonlinear feedback term, leading to no significant change in their results (e.g., p. R128 of Palmer et al. 2014). Therefore, the major features of perturbation transfer and growth at various scales within Lorenz (1969) cannot be interpreted as a BE1 that requires irregular oscillations associated with nonlinear processes. On the other hand, since the Lorenz (1969) linear model provides continual release of instability, the model can effectively reveal the role of a tiny perturbation in triggering the successive growth of systems at various scales (i.e., leading to "chain" reactions that should remain linear). Major features in Lorenz (1969) may be necessary conditions for BE2 but do not necessarily lead to the formation of an organized system (e.g., a tornado or a hurricane) that depends on, among other factors, such as the time required for growing an organized system (depending on the magnitude of the growth rate). More importantly, the 1969 model can represent neither the impact of tiny perturbations at butterfly scales nor their physical processes (e.g., dissipative processes). Thus, as summarized in Table 1, BE2 cannot be revealed using the 1969 model.

In Lorenz (1969), the growth rate was computed for estimating predictability. This approach or similar approaches have been applied in real world models for predictability studies. Since numerical errors may grow faster when a numerical solution has a larger growth rate, it is common to improve model predictability by suppressing "instability" within the model. However, if real world physical instability instead of "numerical" instability is reduced by numerical methods, the model can only simulate a weaker system as compared to the real observed system.

## 3 Conclusions

The Lorenz (1963) model (i.e., 3DLM) with the BE1 has had a large impact on nonlinear/chaotic dynamics and has been extensively studied in physics and applied mathematics (e.g., Smale 1998). The Lorenz (1969) model, with features of energy transfer and a scale dependence of growth rates, has a much deeper impact on approaches for estimating atmospheric predictability using (local) growth rates. Although these models were used to reveal the impact of tiny perturbations, they cannot properly represent tiny perturbations at real butterfly scales and their physical processes. Additionally, limited scale interactions within the nonlinear 3DLM and linearization of the Lorenz (1969) multiscale model disqualified these models for addressing the BE2.

For the BE2, we may still ask how a tiny perturbation (e.g., a butterfly's flap) can or cannot generate an organized system (e.g., a tornado). The BE1 revealed by the 3DLM, with a limited degree of scale interactions, indicate the complexities of solutions within the temporal space and cannot represent the BE2 that requires multiple spatial scale interactions. Within high-dimensional LMs (e.g., the 5DLM) that increase the degree of scale interactions, small-scale processes can introduce a negative nonlinear feedback to suppress, but not enhance, chaotic responses. In comparison, positive feedback associated with a small scale process indicates the importance of the energy source for the small scale process (e.g., Shen 2016). In addition to the energy source and the transfer mechanism that may appear within the Lorenz 1969 model, an ideal model for addressing the BE2 should also include nonlinear intensification and dissipation, which collectively lead to time varying negative and positive feedbacks, and other factors, which include mechanisms for the organization (aggregation) of growing energy. In reality, the energy of butterfly flapping will cascade up and down spatial scales from nonlinear interactions (Pielke 2008, 2013). Downscale propagation will clearly dissipate into molecular motions and heat. Such a model result requires the inclusion of realistic dissipative processes. For upscale energy, less and less energy moves upscale. Thus, for real Earth systems no energy is capable of travelling a long distance (i.e., a large spatial scale) from a small perturbation (at a small temporal and spatial scale). Indeed, once any coherence in energy is lost, no mechanism is present that generates a coherent system (such as a tornado) at a distance. Adding a radiative flux divergence term to the equations discussed in this paper would easily demonstrate this fact. Based on the above analysis of Lorenz's studies and subsequent studies, we conclude that no theoretical or observational evidence indicates the possibility that a butterfly's flap is capable of creating a tornado (i.e., the probability for BE2 is, in reality, zero). Since numerical methods may introduce spurious chaos (Corless 1994) and since numerical models may produce false alarm events, any "numerical" evidence for BE2 is subject to serious verifications versus the nature of weather.

The Lorenz 1963 and 1969 models have been used to reveal the nature of weather with a focus on "chaos". The 3DLM model indeed produces various types of solutions, including steady-state solutions, chaotic solutions, and nonlinear oscillatory solutions (i.e., limit cycle solutions). Linearly unstable solutions are produced by the Lorenz 1969 model. The 3DLM and high-dimensional LMs additionally allow the co-existence of chaotic and non-chaotic solutions. Thus, we propose that the entirety of weather is a superset that consists of both chaotic and non-chaotic processes. Specific weather systems may appear on a chaotic or non-chaotic orbit for a finite period of time. Positive growth rates for a finite period of time may indicate the occurrence of either chaos or (local) instability. Non-periodicity may appear in the presence of chaos or a quasi-periodicity that is associated with two or more incommensurate frequencies. Quasi-periodic flow also never repeats itself.

The above refined view on the nature of weather suggests both potential and challenges. If we can identify non-chaotic solutions such as periodic or quasi-periodic solutions or linearly unstable solutions in advance, we may obtain better predictability. Our future work will focus on improving our understanding of the roles of butterfly effects in real world, high-resolution global models and, thus, our understanding of the conditions under which nonlinear interactions may lead to non-chaotic solutions such as limit cycle solutions and/or chaotic solutions.

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## Appendix A: The Lorenz Model (1963) and a Popular Analogy for Chaos

We first introduce the classical 3DLM, as follows:

$$\frac{dX}{d\tau} = \sigma Y - \sigma X, \qquad (A1)$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \qquad (A2)$$

$$\frac{dZ}{d\tau} = XY - bZ. \qquad (A3)$$

Here, $\tau, \sigma,$ and $r$ represent dimensionless time, the Prandtl number, and the normalized Rayleigh number (or the heating parameter), respectively. Parameter $b$ is a function of the ratio between the vertical scale of the convection cell and its horizontal scale. $(X, Y, Z)$ represent the amplitudes of the three Fourier modes (e.g., Table 1 of Shen 2014). Equations (A1)-(A3) include three types of physical processes, including buoyance/heating, dissipative, and nonlinear processes. The linear buoyance force and the heating force are represented by $\sigma Y$ in Eq. (A1) and $rX$ in Eq. (A2), respectively. The three dissipative terms are $-\sigma X$, $-Y$, and $-bZ$ and are ignored under the dissipationless condition. The two nonlinear terms, $-XZ$ and $XY$, are derived from nonlinear advection of the temperature term in the governing equation for the Rayleigh-Benard convection (RBC; e.g., Saltzmanm 1962; Lorenz 1963; Eq. (2) of Shen 2014). With the exception of the heating parameter (r), the following parameters are kept constant: $\sigma = 10$ and $b = 8/3$. Since climate and weather involve open systems, an assumption of constant parameters in numerical simulations using the 3DLM is not realistic and, thus, the applicability of numerical results to climate or weather should be interpreted with caution.

The sensitive dependence on initial conditions (i.e., the BE1) has been illustrated using the following folklore (e.g., Gleick, 1987; Drazin, 1992):

*"For want of a nail, the shoe was lost.*
*For want of a shoe, the horse was lost.*
*For want of a horse, the rider was lost.*
*For want of a rider, the battle was lost.*
*For want of a battle, the kingdom was lost.*
*And all for the want of a horseshoe nail."*

However, in 2008, Prof. Lorenz stated that *he did not feel that this verse described true chaos but that it better illustrated the simpler phenomenon of instability; and that the verse implicitly suggests that subsequent small events will not reverse the outcome* (Lorenz, 2008). In other words, the verse only indicates divergence, not boundedness. Boundedness is important for the finite size of a butterfly pattern. Additionally, the verse does not consider any (future) possibility for a (small-scale) process to bring a negative feedback, as illustrated using high-dimensional Lorenz models.

Table 1: Definitions of BE1 and BE2 and characteristics of Lorenz models.

| **Original source** | **Lorenz 1963** | **Lorenz 1972/1969** |
|---|---|---|
| Classification of butterfly effects | BE1 | BE2 |
| Mathematical or physical definitions | Sensitive dependence on ICs | The enabling role of a tiny perturbation in producing an organized large-scale system |
| Physical processes in PDEs | Nonlinear advection, heating/buoyance, and dissipation | Nonlinear advection (in the Lorenz 1969 model) |
| ODEs | Nonlinear | Linearized with realistic basic winds |
| Impact of initial perturbations on numerical solutions at system allowed scales | Changing the time sequence of events | Triggering modes with constant growth rates at various scales |
| Impact of additional small-scale processes in the revised models | Providing negative and/or positive feedback (e.g., Shen, 2014, 2015a,b) or producing additional oscillatory components (e.g., Faghih-Naini and Shen 2018) | Producing different growth rates (e.g., Rotunno and Snyder 2008) |
| Growth rates | Local time-varying; one positive Lyapunov exponent (i.e., constant long-time average divergence) | Constant in time; dependence on spatial scales |
| Perturbations at butterfly (spatial) scales | Not represented in the IC or equations* | Not represented in the IC or equations |
| Remark | • The word "butterfly" of BE1 mainly represents the system (e.g., the butterfly pattern solution);<br>• Tiny perturbations in the time-varying amplitudes of ICs are often viewed as butterfly flaps; such perturbations appear at the system's spatial scales and may change the time sequence of events at system scales;<br>• All of the time-varying events are determined by the system whose solutions are sensitive to internal changes in ICs. | • The "butterfly" of BE2 means the tiny perturbation at real butterfly scales (i.e., small amplitudes and spatial scales);<br>• Such "small" processes are not represented in the ICs or models;<br>• Linearized models are not suitable for addressing the physical processes of BE2. |

*If the three-mode 3DLM includes perturbations at butterfly scales, it cannot include other systems at intermediate or large scales.

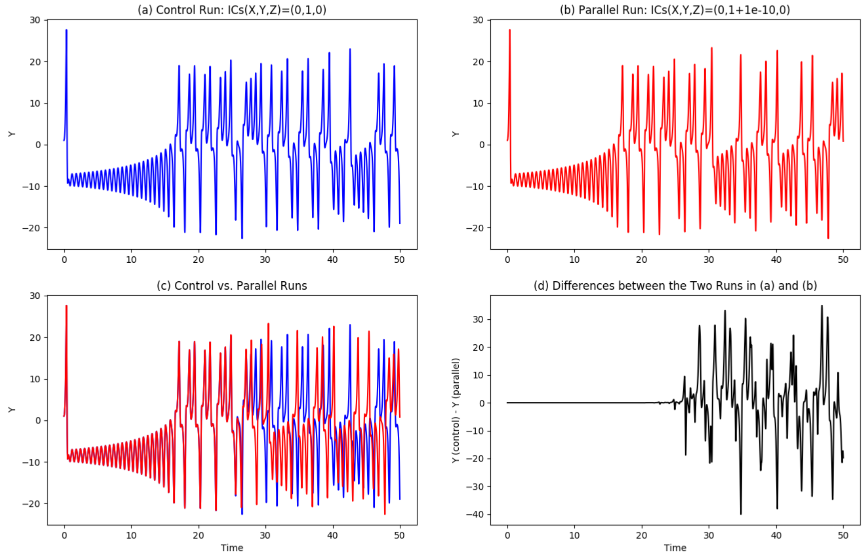


*Figure 1: An illustration of the bounded divergence of two nearby trajectories within the 3DLM with r = 28 and* $\sigma = 10$. *Panels (a) and (b) display solutions from the control and parallel runs, respectively, the latter of which adds a small perturbation (*$1e^{-10}$*) into the initial value of Y. Panel (c) reveals the sensitive dependence of solutions on the initial conditions. Panel (d) displays bounded differences in solutions for the control and parallel runs.*

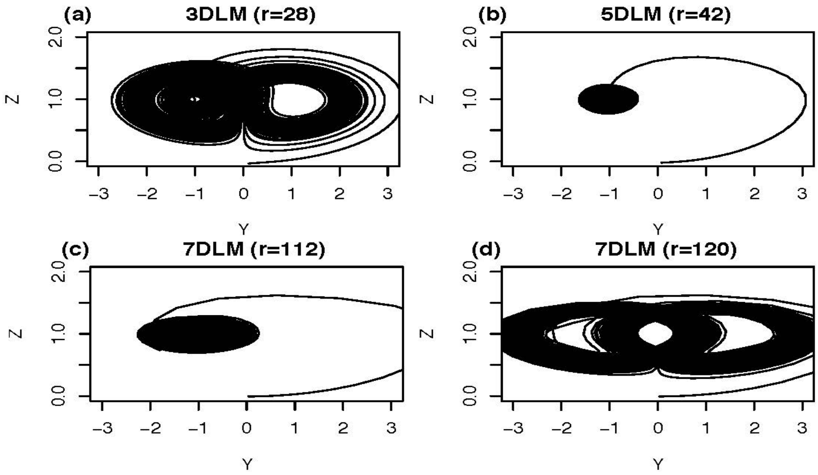


*Figure 2: Phase space plots for (Y, Z) in various LMs. (a) Strange attractors with r = 28 within the 3DLM that display the well-known butterfly pattern. (b) A stable solution with r = 42 within the 5DLM. (c-d) Stable and chaotic solutions with r = 112 and r = 120 within the 7DLM, respectively. Here, r is the normalized Rayleigh parameter. The figure indicates that high-dimensional LMs with the proper inclusion of new modes are more predictable than the 3DLM. Data are from Shen (2016).*

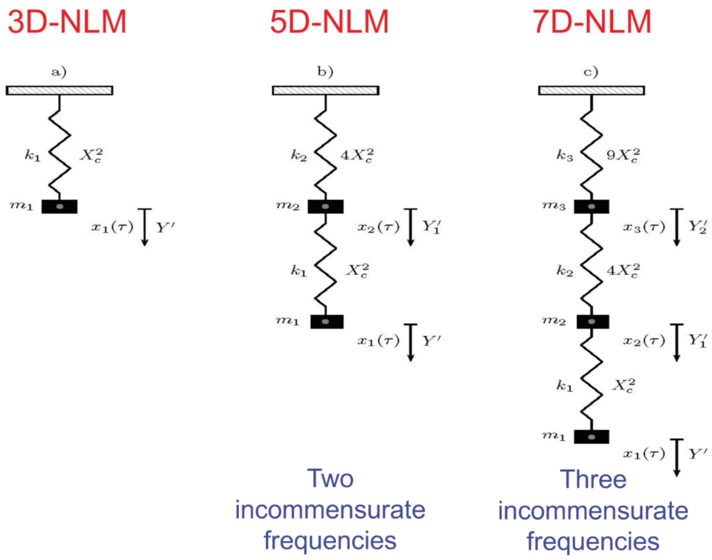


*Figure 3: Systems with one mass and one spring (a), two masses and two springs (b), and three masses and three springs (c). The three masses are identical (i.e., $m_1 = m_2 = m_3$). The three spring constants, $k_1$, $k_2$, and $k_3$, are $X_c^2$, $4X_c^2$, and $9X_c^2$, respectively. The governing equations for the above systems in panels (a)-(c) are identical to those for the locally linear 3D, 5D, and 7D non-dissipative Lorenz models (NLMs), respectively. The comparison illustrates how a nonlinear feedback loop and its extension, enabled by the proper selection of high wavenumber modes, can produce recurrent (i.e., periodic or quasi-periodic) solutions.*